Finite-time future singularities and cosmologies in modified gravity

Main references:

[1] K. Bamba, S. Nojiri, S. D. Odintsov and M. Sasaki,
Gen. Relativ. Gravit. 44, 1321 (2012) [arXiv:1104.2692 [hep-th]].

[2] K. Bamba, R. Myrzakulov, S. Nojiri and S. D. Odintsov,
Phys. Rev. D 85, 104036 (2012) [arXiv:1202.4057 [gr-qc]].

Kobayashi-Maskawa Institute
for the Origin of Particles and the Universe

3rd International Workshop on Dark Matter,
Dark Energy and Matter-Antimatter Asymmetry,
NCTS, National Tsing Hua University,
28th December, 2012

Presenter : **Kazuharu Bamba** (*KMI, Nagoya University*)

Collaborators : **Ratbay Myrzakulov** (*EICTP, Eurasian National University*),
Shin'ichi Nojiri (*KMI and Dep. of Physics, Nagoya University*),
Sergei D. Odintsov (*ICREA and IEEC-CSIC*),
Misao Sasaki (*YITP, Kyoto University and KIAS*)

< Contents >

I. Introduction

2011 Nobel Prize in Physics

- Recent observations of Supernova (SN) Ia confirmed that the current expansion of the universe is accelerating.

[Perlmutter *et al*. [Supernova Cosmology Project Collaboration], Astrophys. J. 517, 565 (1999)] [Riess *et al*. [Supernova Search Team Collaboration], Astron. J. 116, 1009 (1998)]
[Astier *et al*. [The SNLS Collaboration], Astron. Astrophys. 447, 31 (2006)]

- There are two approaches to explain the current cosmic acceleration. **[Copeland, Sami and Tsujikawa, Int. J. Mod. Phys. D 15, 1753 (2006)]**

[Caldwell and Kamionkowski, Ann. Rev. Nucl. Part. Sci. 59, 397 (2009)]
[Amendola and Tsujikawa, *Dark Energy* (Cambridge University press, 2010)]
[Li, Li, Wang and Wang, Commun. Theor. Phys. 56, 525 (2011)]
[KB, Capozziello, Nojiri and Odintsov, Astrophys. Space Sci. 342, 155 (2012)]

< Gravitational field equation >

$$G_{\mu\nu} = \kappa^2 T_{\mu\nu}$$

Gravity **Matter**

$G_{\mu\nu}$: Einstein tensor

$T_{\mu\nu}$: Energy-momentum tensor

$\kappa^2 \equiv 8\pi / {M_{\mathrm{Pl}}}^2$, M_{Pl} : Planck mass

(1) **General relativistic approach** ⟶ **Dark Energy**

(2) **Extension of gravitational theory**

(1) General relativistic approach

- **Cosmological constant**
- **Scalar field : x-matter, Quintessence** ← Canonical field

[Chiba, Sugiyama and Nakamura, Mon. Not. Roy. Astron. Soc. 289, L5 (1997)]
[Caldwell, Dave and Steinhardt, Phys. Rev. Lett. 80, 1582 (1998)]

Cf. Pioneering work: [Fujii, Phys. Rev. D 26, 2580 (1982)]

Phantom ← Wrong sign kinetic term

[Caldwell, Phys. Lett. B 545, 23 (2002)]

K-essence ← Non canonical kinetic term

[Chiba, Okabe and Yamaguchi, Phys. Rev. D 62, 023511 (2000)]
[Armendariz-Picon, Mukhanov and Steinhardt, Phys. Rev. Lett. 85, 4438 (2000)]

Tachyon ← String theories * The mass squared is negative.

[Padmanabhan, Phys. Rev. D 66, 021301 (2002)]

- **Fluid :(Generalized) Chaplygin gas**

$A > 0,\ u$: Constants
ρ : Energy density
P : Pressure

Equation of state (EoS): $P = -A/\rho^u$

[Kamenshchik, Moschella and Pasquier, Phys. Lett. B 511, 265 (2001)] ← $(u = 1)$
[Bento, Bertolami and Sen, Phys. Rev. D 66, 043507 (2002)]

(2) Extension of gravitational theory

Cf. Application to inflation:
[Starobinsky, Phys. Lett. B 91, 99 (1980)]

- ***F*(*R*) gravity**

$F(R)$: Arbitrary function of the Ricci scalar R

[Capozziello, Cardone, Carloni and Troisi, Int. J. Mod. Phys. D 12, 1969 (2003)]

[Carroll, Duvvuri, Trodden and Turner, Phys. Rev. D 70, 043528 (2004)]

[Nojiri and Odintsov, Phys. Rev. D 68, 123512 (2003)]

$f_i(\phi)$ $(i=1,2)$: Arbitrary function of a scalar field ϕ

- **Scalar-tensor theories** ← $f_1(\phi)R$

[Boisseau, Esposito-Farese, Polarski and Starobinsky, Phys. Rev. Lett. 85, 2236 (2000)]

[Gannouji, Polarski, Ranquet and Starobinsky, JCAP 0609, 016 (2006)]

- **Ghost condensates** **[Arkani-Hamed, Cheng, Luty and Mukohyama, JHEP 0405, 074 (2004)]**

- **Higher-order curvature term**

Gauss-Bonnet term with a coupling to a scalar field: $f_2(\phi)\mathcal{G}$

$$\mathcal{G} \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$$

[Nojiri, Odintsov and Sasaki, Phys. Rev. D 71, 123509 (2005)]

$R_{\mu\nu}$: Ricci curvature tensor

$R_{\mu\nu\rho\sigma}$: Riemann tensor

- **$f(\mathcal{G})$ gravity** ← $\frac{R}{2\kappa^2} + f(\mathcal{G})$

$\kappa^2 \equiv 8\pi G$

G : Gravitational constant

[Nojiri and Odintsov, Phys. Lett. B 631, 1 (2005)]

- **DGP braneworld scenario**

  [Dvali, Gabadadze and Porrati, Phys. Lett B 485, 208 (2000)]

  [Deffayet, Dvali and Gabadadze, Phys. Rev. D 65, 044023 (2002)]

- **Non-local gravity** ← Quantum effects **[Deser and Woodard, Phys. Rev. Lett. 99, 111301 (2007)]**

  [Nojiri and Odintsov, Phys. Lett. B 659, 821 (2008)]

- ***f*(*T*) gravity** : Extended teleparallel Lagrangian described by the torsion scalar T.

  [Bengochea and Ferraro, Phys. Rev. D 79, 124019 (2009)]

  [Linder, Phys. Rev. D 81, 127301 (2010) [Erratum-ibid. D 82, 109902 (2010)]]

  - “Teleparallelism” : One could use the Weitzenböck connection, which has no curvature but torsion, rather than the curvature defined by the Levi-Civita connection.

  [Hayashi and Shirafuji, Phys. Rev. D 19, 3524 (1979) [Addendum-ibid. D 24, 3312 (1982)]]

- **Galileon gravity** **[Nicolis, Rattazzi and Trincherini, Phys. Rev. D 79, 064036 (2009)]**

  $\Box\phi(\partial^{\mu}\phi\partial_{\mu}\phi)$ ← Longitudinal graviton (a branebending mode ϕ)

  - The equations of motion are invariant under the Galilean shift: $\partial_{\mu}\phi \rightarrow \partial_{\mu}\phi + b_{\mu}$

    ⇨ One can keep the equations of motion up to the second-order.

    → This property is welcome to avoid the appearance of an extra degree of freedom associated with ghosts.

  $\Box$: Covariant d'Alembertian

- **Massive gravity** **[de Rham and Gabadadze, Phys. Rev. D 82, 044020 (2010)]**

  [de Rham and Gabadadze and Tolley, Phys. Rev. Lett. 106, 231101 (2011)]

  Review: [Hinterbichler, Rev. Mod. Phys. 84, 671 (2012)]

< Flat Friedmann-Lemaître-Robertson-Walker (FLRW) space-time >

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)d\boldsymbol{x}^2$$

$a(t)$: Scale factor

< Equation for $a(t)$ with a perfect fluid >

$$\frac{\ddot{a}}{a} = -\frac{\kappa^2}{6}\underline{(1+3w)\,\rho}$$

$$T^\mu_\nu = \mathrm{diag}(-\rho, P, P, P)$$

ρ : Energy density

P : Pressure

$\dot{} = \partial/\partial t$

$\boxed{w \equiv \frac{P}{\rho}}$ **: Equation of state (EoS)**

$\ddot{a} > 0$ **: Accelerated expansion**

⟹ $\boxed{w < -\frac{1}{3}}$: **Condition for accelerated expansion**

Cf. Cosmological constant ⟹ $w = -1$

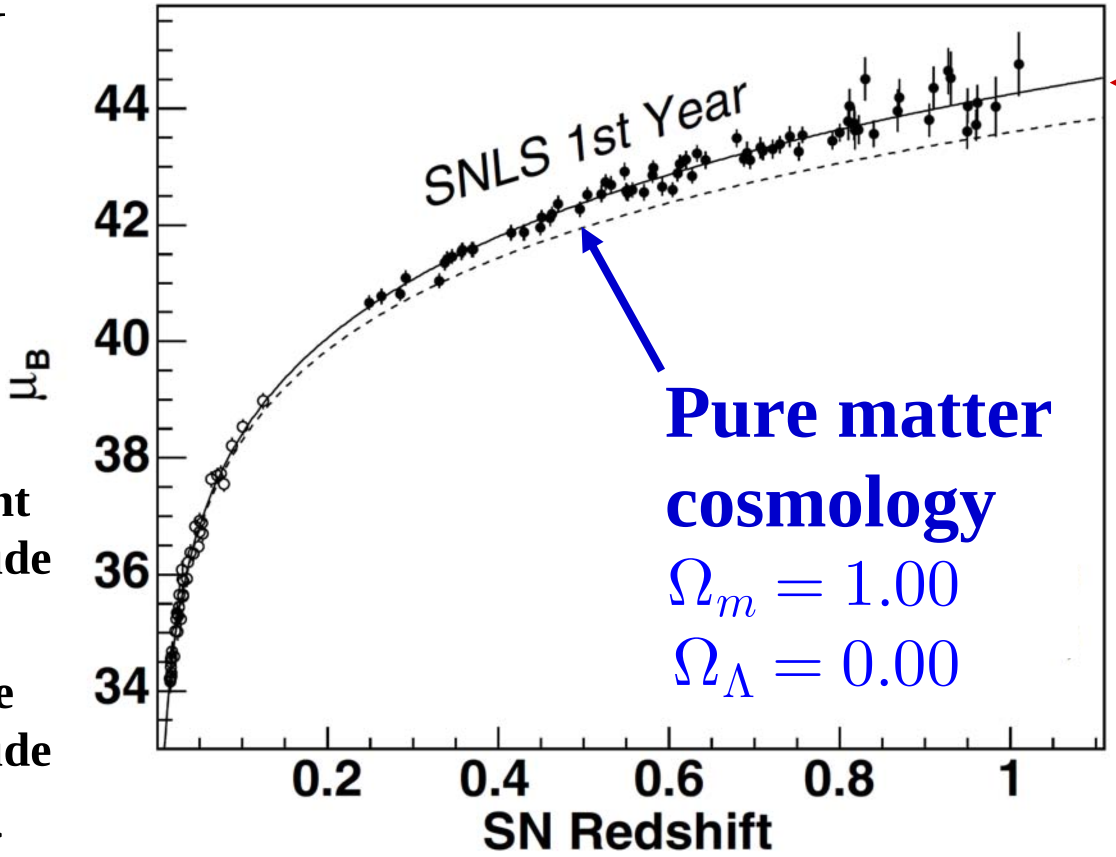
< SNLS data >

$m - M$

: Distance estimator

m **: Apparent magnitude**

M **: Absolute magnitude**



z

Flat Λ cosmology

$\Omega_m = 0.26$

$\Omega_\Lambda = 0.74$

Pure matter cosmology

$\Omega_m = 1.00$

$\Omega_\Lambda = 0.00$

< Fig. 1 >

From [Astier *et al.* [The SNLS Collaboration], Astron. Astrophys. 447, 31 (2006)].

$$\frac{1}{H_0^2}\frac{\ddot{a}}{a} = -\frac{\Omega_m}{2}(1+z)^3 + \Omega_\Lambda$$

$$\Omega_m \equiv \frac{\kappa^2 \rho(t_0)}{3H_0^2}$$: Density parameter for matter

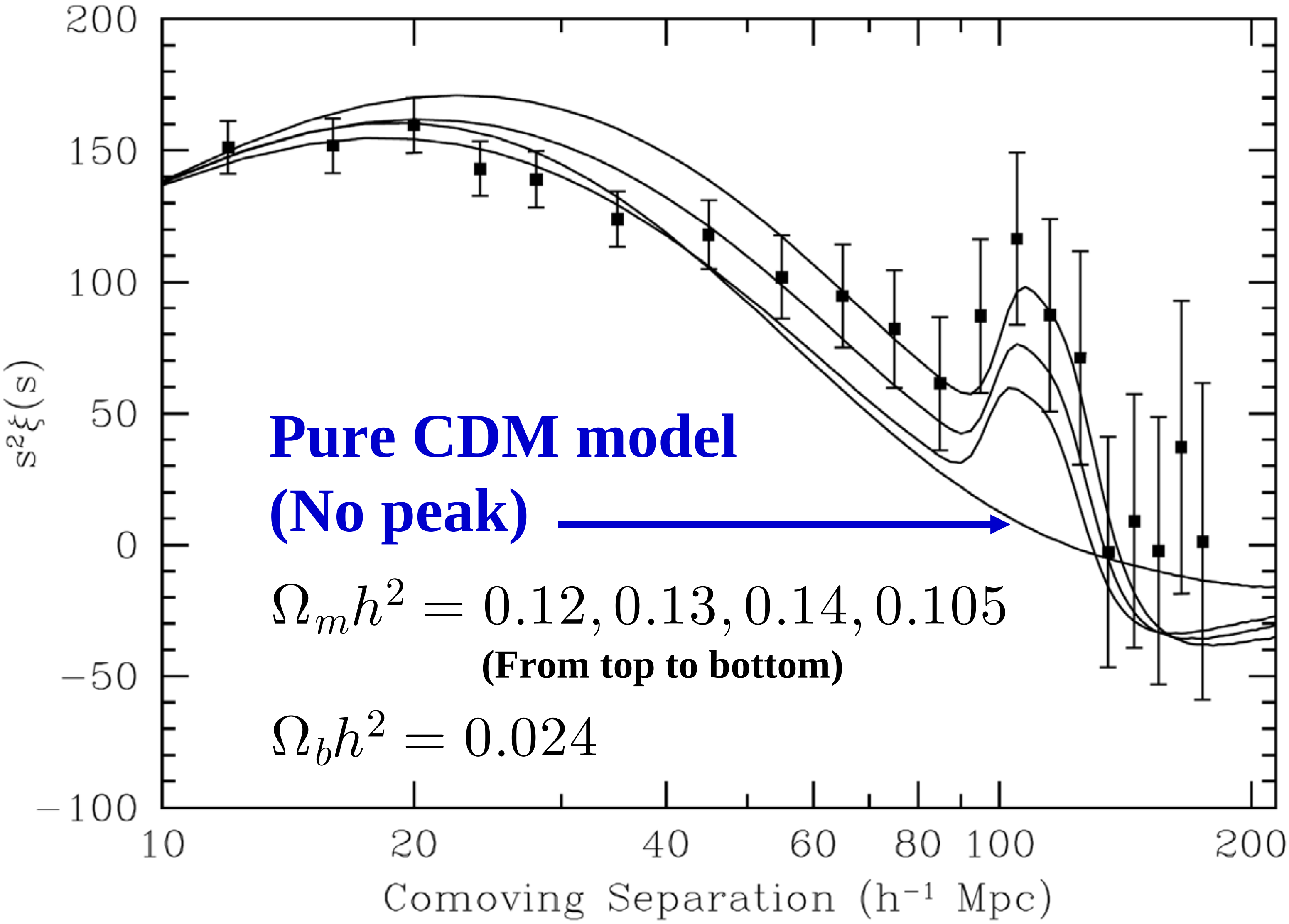
$$\Omega_\Lambda \equiv \frac{\Lambda}{3H_0^2}$$: Density parameter for Λ

$a_0 = 1$

$1 + z = \frac{a_0}{a}$, z : Redshift

"0" denotes quantities at the present time t_0 .

< Baryon acoustic oscillation (BAO) >



Pure CDM model
(No peak)

$\Omega_m h^2 = 0.12, 0.13, 0.14, 0.105$

(From top to bottom)

$\Omega_b h^2 = 0.024$

Special pattern in the large-scale correlation function of Sloan Digital Sky Survey (SDSS) luminous red galaxies

< Fig. 2 >

From [Eisenstein *et al*. [SDSS Collaboration], Astrophys. J. 633, 560 (2005)].

< 9-year WMAP data on the current value of w >

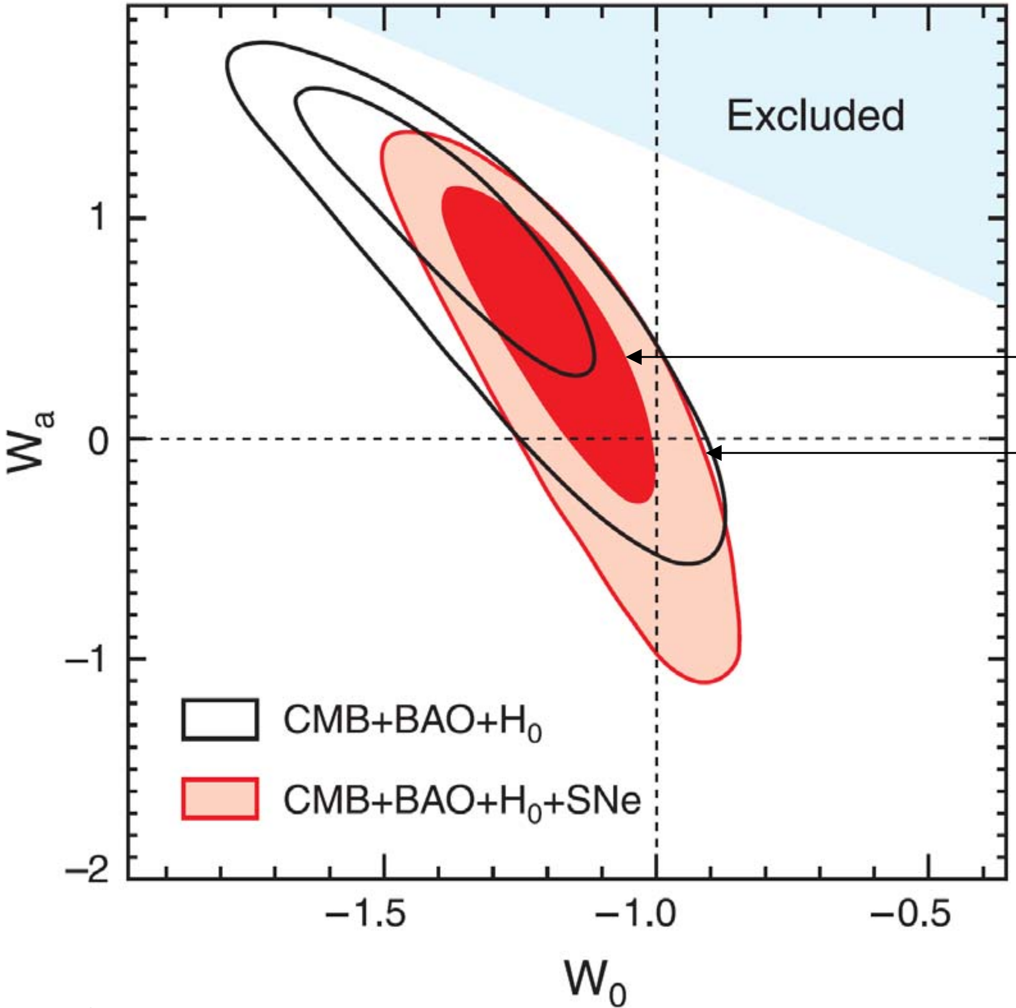
[Hinshaw *et al.*, arXiv:1212.5226 [astro-ph.CO]]

Hubble constant (H_0) measurement

- For constant w :

$$w = \begin{cases} -1.084 \pm 0.063 & \text{(flat)} \\ -1.122^{+0.068}_{-0.067} & \text{(non-flat)} \end{cases} \quad (68\%\ \mathrm{CL})$$

(From $WMAP$+eCMB+BAO+H_0+SNe.)



From [Hinshaw *et al.*, arXiv:1212.5226 [astro-ph.CO]].

(68% CL)

(95% CL)

Time-dependent w

$$w(a) = w_0 + w_a(1 - a)$$

$a = \frac{1}{1+z}$

w_0 : Current value of w

z : Redshift

< Fig. 3 >

(From $WMAP$+eCMB +BAO+H_0+SNe.)

- For the flat universe, a variable EoS :

$$\boxed{w_0 = -1.17^{+0.13}_{-0.12} \, , \; w_a = 0.35^{+0.50}_{-0.49} \; (68\% \text{ CL})}$$

→ It is meaningful to investigate theoretical features of modified gravity theories.

We concentrate on the existence of finite time future singularities in (i) Non-local gravity and (ii) $f(T)$ gravity.

- It is known that the finite-time future singularities can be classified in the following manner:

t_{s} : Time when finite-time future singularities appear

Type I ("Big Rip"):	In the limit $t \to t_{\mathrm{s}}$, $\underline{a \longrightarrow \infty\,, \quad \rho_{\mathrm{eff}} \to \infty\,, \quad \lvert P_{\mathrm{eff}} \rvert \to \infty}$ ∗ The case in which ρ_{eff} and P_{eff} becomes finite values at $t = t_{\mathrm{s}}$ is also included.
Type II ("sudden"):	$a \longrightarrow a_{\mathrm{s}}\,, \quad \rho_{\mathrm{eff}} \to \rho_{\mathrm{s}}\,, \quad \underline{\lvert P_{\mathrm{eff}} \rvert \to \infty}$
Type III:	$a \longrightarrow a_{\mathrm{s}}\,, \quad \underline{\rho_{\mathrm{eff}} \to \infty\,, \quad \lvert P_{\mathrm{eff}} \rvert \to \infty}$
Type IV: **[Nojiri, Odintsov and Tsujikawa, Phys. Rev. D 71, 063004 (2005)]**	$a \longrightarrow a_{\mathrm{s}}\,, \quad \rho_{\mathrm{eff}} \to 0\,, \quad \lvert P_{\mathrm{eff}} \rvert \to 0$ ∗ Higher derivatives of H diverge. ∗ The case in which ρ_{eff} and/or $\lvert P_{\mathrm{eff}} \rvert$ asymptotically approach finite values is also included.

II. Finite-time future singularities in non-local gravity

Non-local gravity ⇐ **produced by quantum effects**

[Deser and Woodard, Phys. Rev. Lett. 99, 111301 (2007)]
[Nojiri and Odintsov, Phys. Lett. B 659, 821 (2008)]

→ This theory can explain the current accelerated expansion of the universe.

- It is known that so-called matter instability occurs in $F(R)$ gravity.

[Dolgov and Kawasaki, Phys. Lett. B 573, 1 (2003)]

→ This implies that the curvature inside matter sphere becomes very large and hence the curvature singularity could appear.

[Arbuzova and Dolgov, Phys. Lett. B 700, 289 (2011)]
Cf. [KB, Nojiri and Odintsov, Phys. Lett. B 698, 451 (2011)]

⇒ It is important to examine whether there exists the curvature singularity, i.e., “**the finite-time future singularities**” **in non-local gravity**.

A. Non-local gravity

$g = \det(g_{\mu\nu})$ $g_{\mu\nu}$: Metric tensor

f : Some function Λ : Cosmological constant

< Action >

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}\left[R\left(1+\underline{f(\Box^{-1}R)}\right)-2\Lambda\right]+\mathcal{L}_{\mathrm{matter}}\left(Q;g\right)\right\}$$

Non-local gravity

By introducing two scalar fields η and ξ , we find

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}\left[R\left(1+f(\eta)\right)+\xi\,\underline{\left(\Box\eta - R\right)}-2\Lambda\right]+\mathcal{L}_{\mathrm{matter}}\right\}$$

$$= \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}\left[R\left(1+f(\eta)\right)-\partial_\mu\xi\partial^\mu\eta-\xi R-2\Lambda\right]+\mathcal{L}_{\mathrm{matter}}\right\}$$

- By the variation of the action in the first expression over ξ , we obtain

$\underline{\Box\eta = R}$ (or $\eta = \Box^{-1}R$)

→ Substituting this equation into the action in the first expression, one re-obtains the starting action.

∇_μ : Covariant derivative operator

$\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$

: Covariant d'Alembertian

$\mathcal{L}_{\mathrm{matter}}\left(Q;g\right)$

: Matter Lagrangian

Q : Matter fields

< Gravitational field equation >

$$0 = \frac{1}{2} g_{\mu\nu} \left[R \left(1 + f(\eta) - \xi\right) - \partial_\rho \xi \partial^\rho \eta - 2\Lambda \right] - R_{\mu\nu} \left(1 + f(\eta) - \xi\right)$$
$$+ \frac{1}{2} \left(\partial_\mu \xi \partial_\nu \eta + \partial_\mu \eta \partial_\nu \xi \right) - \left(g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu \right) \left(f(\eta) - \xi \right) + \kappa^2 T_{\mathrm{matter}\,\mu\nu}$$

$$T_{\mathrm{matter}\,\mu\nu} \equiv -\left(2/\sqrt{-g}\right) \left(\delta \sqrt{-g} \mathcal{L}_{\mathrm{matter}} / \delta g^{\mu\nu} \right)$$

: Energy-momentum tensor of matter

- The variation of the action with respect to η gives

$$0 = \Box \xi + f'(\eta) R$$

$'$ (prime) : Derivative with respect to η

< Flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric >

$$ds^2 = -dt^2 + a^2(t) \sum_{i=1,2,3} \left(dx^i \right)^2$$

$a(t)$: Scale factor

- We consider the case in which the scalar fields η and ξ only depend on time.

→ Gravitational field equations in the flat FLRW background:

$$0=-3H^2\left(1+f(\eta)-\xi\right)+\frac{1}{2}\dot{\xi}\dot{\eta}-3H\left(f'(\eta)\dot{\eta}-\dot{\xi}\right)+\Lambda+\kappa^2\rho_{\mathrm{m}}$$

$$0=\left(2\dot{H}+3H^2\right)\left(1+f(\eta)-\xi\right)+\frac{1}{2}\dot{\xi}\dot{\eta}+\left(\frac{d^2}{dt^2}+2H\frac{d}{dt}\right)\left(f(\eta)-\xi\right)-\Lambda+\kappa^2P_{\mathrm{m}}$$

$\dot{}=\partial/\partial t$ $\quad H=\dot{a}/a$: Hubble parameter

ρ_{m} and P_{m} : Energy density and pressure of matter, respectively.

→ For a perfect fluid of matter: $T_{\mathrm{matter}\,00}=\rho_{\mathrm{m}}$

$T_{\mathrm{matter}\,ij}=P_{\mathrm{m}}\delta_{ij}$

< Equations of motion for η and ξ >

$$0=\ddot{\eta}+3H\dot{\eta}+6\dot{H}+12H^2$$

$$0=\ddot{\xi}+3H\dot{\xi}-\left(6\dot{H}+12H^2\right)f'(\eta)$$

$$R=6\dot{H}+12H^2$$

B. Finite-time future singularities

→ We analyze an asymptotic solution of the gravitational field equations in the limit of $t \longrightarrow t_{\mathrm{s}}$.

- We consider the case in which the Hubble parameter is expressed as

$$\boxed{H \sim \frac{h_{\mathrm{s}}}{(t_{\mathrm{s}} - t)^q}}$$

h_{s} : Positive constant

q : Non-zero constant larger than -1 $(q > -1, q \neq 0)$

∗ We only consider the period $0 < t < t_{\mathrm{s}}$.

- When $t \longrightarrow t_{\mathrm{s}}$, $R = 6\dot{H} + 12H^2 \longrightarrow \infty$

Scale factor

$$\boxed{a \sim a_{\mathrm{s}} \exp\left[\frac{h_{\mathrm{s}}}{q-1}(t_{\mathrm{s}} - t)^{-(q-1)}\right]}$$

a_{s} : Constant

- We take a form of $f(\eta)$ as $\boxed{f(\eta) = f_{\mathrm{s}}\eta^{\sigma}}$.

$f_{\mathrm{s}}(\neq 0)$, $\sigma(\neq 0)$: Non-zero constants

⇨ We acquire the integration forms of η and ξ .

η_{C}, ξ_{C} : Integration constants

- We examine the behavior of each term of the gravitational field equations in the limit $t \to t_{\mathrm{s}}$, in particular that of the leading terms.

→ We study the condition that the leading term vanishes in both gravitational field equations and hence an asymptotic solution can be obtained.

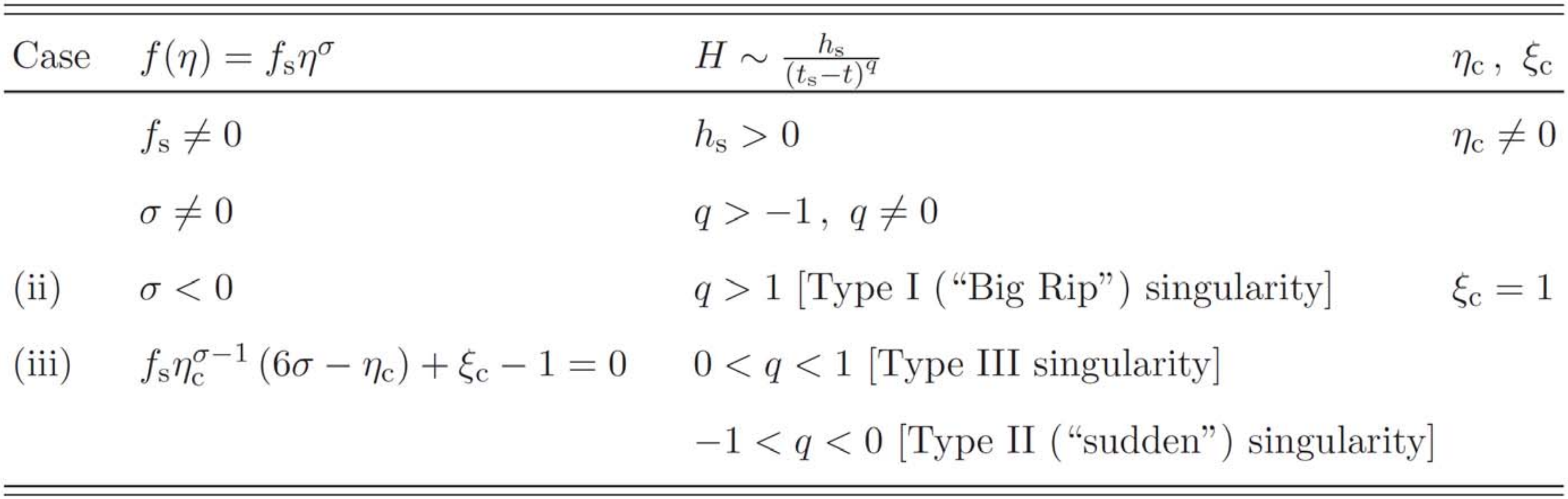
- The finite-time future singularities described by the expression of H in non-local gravity have the following properties:

$$H \sim \frac{h_{\mathrm{s}}}{(t_{\mathrm{s}}-t)^q}$$

For $q > 0$,	Type I (“Big Rip”)
For $-1 < q < 0$,	Type II (“sudden”)
For $q > -1$,	Type III

< Table 1 >

Range and conditions for the value of parameters of $f(\eta)$, H , and η_{c} and ξ_{c} in order that the finite-time future singularities can exist.



Case	$f(\eta) = f_{\mathrm{s}}\eta^{\sigma}$	$H \sim \frac{h_{\mathrm{s}}}{(t_{\mathrm{s}}-t)^q}$	η_{c} , ξ_{c}
	$f_{\mathrm{s}} \neq 0$	$h_{\mathrm{s}} > 0$	$\eta_{\mathrm{c}} \neq 0$
	$\sigma \neq 0$	$q > -1$, $q \neq 0$	
(ii)	$\sigma < 0$	$q > 1$ [Type I (“Big Rip”) singularity]	$\xi_{\mathrm{c}} = 1$
(iii)	$f_{\mathrm{s}}\eta_{\mathrm{c}}^{\sigma-1}(6\sigma - \eta_{\mathrm{c}}) + \xi_{\mathrm{c}} - 1 = 0$	$0 < q < 1$ [Type III singularity]	
		$-1 < q < 0$ [Type II (“sudden”) singularity]	

III. Finite-time future singularities in *f*(*T*) gravity

< Formulations in teleparallelism >

- $g_{\mu\nu} = \eta_{AB} e^A_\mu e^B_\nu$
- $T^\rho{}_{\mu\nu} \equiv e^\rho_A \left(\partial_\mu e^A_\nu - \partial_\nu e^A_\mu\right)$

  : Torsion tensor
- $K^{\mu\nu}{}_\rho \equiv -\frac{1}{2}\left(T^{\mu\nu}{}_\rho - T^{\nu\mu}{}_\rho - T_\rho{}^{\mu\nu}\right)$

  : Contorsion tensor
- $\boxed{T \equiv S_\rho{}^{\mu\nu} T^\rho{}_{\mu\nu}}$: **Torsion scalar**

$$S_\rho{}^{\mu\nu} \equiv \frac{1}{2}\left(K^{\mu\nu}{}_\rho + \delta^\mu_\rho\, T^{\alpha\nu}{}_\alpha - \delta^\nu_\rho\, T^{\alpha\mu}{}_\alpha\right)$$

η_{AB} : Minkowski metric

$e_A(x^\mu)$: Orthonormal tetrad components

* An index A runs over 0, 1, 2, 3 for the tangent space at each point of x^μ the manifold.

* μ and ν are coordinate indices on the manifold and also run over 0, 1, 2, 3, and $e_A(x^\mu)$ forms the tangent vector of the manifold.

Instead of the Ricci scalar R for the Lagrangian density in general relativity, the teleparallel Lagrangian density is described by the torsion scalar T.

< Modified teleparallel action for *f*(*T*) theory >

Action

$$\boxed{S = \int d^4x|e|\left[\frac{f(T)}{\underline{2\kappa^2}} + \mathcal{L}_{\mathrm{M}}\right]}$$

$$|e| = \det\left(e^A_\mu\right) = \sqrt{-g}$$

$\mathcal{L}_{\mathrm{M}}$: Matter Lagrangian

${T^{(\mathrm{M})}}_\rho{}^\nu$: Energy-momentum tensor of matter

→ **Gravitational field equation**

[Bengochea and Ferraro, Phys. Rev. D 79, 124019 (2009)]

$$\frac{1}{e}\partial_\mu\left(eS_A{}^{\mu\nu}\right)f' - e_A^\lambda T^\rho{}_{\mu\lambda}S_\rho{}^{\nu\mu}f' + S_A{}^{\mu\nu}\partial_\mu\left(T\right)f'' + \frac{1}{4}e_A^\nu f = \frac{\kappa^2}{2}e_A^\rho T^{(\mathrm{M})}{}_\rho{}^\nu$$

* A prime denotes a derivative with respect to T.

⇨ **The gravitational field equation in *f*(*T*) gravity is 2nd order, although it is 4th order in *F*(*R*) gravity.**

- We assume the flat FLRW space-time with the metric,

$$ds^2 = dt^2 - a^2(t)d\boldsymbol{x}^2 \Longrightarrow \boxed{T = -6H^2}$$

$$g_{\mu\nu} = \mathrm{diag}(1, -a^2, -a^2, -a^2), \quad e^A_\mu = (1, a, a, a)$$

< Finite-time future singularities >

→ Gravitational field equations in the flat FLRW background

$$\boxed{H^2 = \frac{\kappa^2}{3}\left(\rho_{\mathrm{M}} + \rho_{\mathrm{DE}}\right) , \qquad \dot{H} = -\frac{\kappa^2}{2}\left(\rho_{\mathrm{M}} + P_{\mathrm{M}} + \rho_{\mathrm{DE}} + P_{\mathrm{DE}}\right)}$$

$$\rho_{\mathrm{DE}} = \frac{1}{2\kappa^2} J_1 \qquad J_1 \equiv -T - f + 2TF \qquad F \equiv df/dT$$

$$P_{\mathrm{DE}} = -\frac{1}{2\kappa^2}\left(4J_2 + J_1\right) \qquad J_2 \equiv \left(1 - F - 2TF'\right)\dot{H} \qquad F' = dF/dT$$

- **Continuity equation** $\dot{\rho}_{\mathrm{DE}} + 3H\left(\rho_{\mathrm{DE}} + P_{\mathrm{DE}}\right) = 0$

- **Effective EoS**

$$\boxed{w_{\mathrm{eff}} \equiv \frac{P_{\mathrm{eff}}}{\rho_{\mathrm{eff}}} = -1 - \frac{2\dot{H}}{3H^2}} \quad \Rightarrow \quad \boxed{w_{\mathrm{eff}} \approx w_{\mathrm{DE}} = P_{\mathrm{DE}}/\rho_{\mathrm{DE}}}$$

$$\rho_{\mathrm{eff}} \equiv \frac{3H^2}{\kappa^2} , \quad P_{\mathrm{eff}} \equiv -\frac{2\dot{H} + 3H^2}{\kappa^2}$$

$$\rho_{\mathrm{DE}} \gg \rho_{\mathrm{M}} , \quad P_{\mathrm{DE}} \gg P_{\mathrm{M}}$$

$$\rho_{\mathrm{eff}} \approx \rho_{\mathrm{DE}} , \quad P_{\mathrm{eff}} \approx P_{\mathrm{DE}}$$

- **Hubble parameter**

$$H \sim \frac{h_{\mathrm{s}}}{(t_{\mathrm{s}} - t)^q} \qquad \text{for} \;\; q > 0$$

$$H \sim H_{\mathrm{s}} + \frac{h_{\mathrm{s}}}{(t_{\mathrm{s}} - t)^q} \qquad \text{for} \;\; q < -1 \,, \;\; -1 < q < 0$$

$$h_{\mathrm{s}}(> 0), \quad q(\neq 0, \, -1) \,, \;\; H_{\mathrm{s}}(> 0)$$

Scale factor

⇨

$$a \sim a_{\mathrm{s}} \exp \left[\frac{h_{\mathrm{s}}}{q-1} (t_{\mathrm{s}} - t)^{-(q-1)} \right] \qquad \text{for} \;\; 0 < q < 1 \,, \;\; 1 < q$$

$$a \sim a_{\mathrm{s}} \frac{h_{\mathrm{s}}}{(t_{\mathrm{s}} - t)^{h_{\mathrm{s}}}} \qquad \text{for} \;\; q = 1$$

< Table 2 >

Conditions to produce the finite-time future singularities in the limit of $t \longrightarrow t_{\mathrm{s}}$.

$q(\neq 0,\ -1)$	$H\ (t \to t_{\mathrm{s}})$	$\dot{H}\ (t \to t_{\mathrm{s}})$	ρ_{DE}	P_{DE}
$q \geq 1$ [Type I (“Big Rip”) singularity]	$H \to \infty$	$\dot{H} \to \infty$	$J_1 \neq 0$	$J_1 \neq 0$ or $J_2 \neq 0$
$0 < q < 1$ [Type III singularity]	$H \to \infty$	$\dot{H} \to \infty$	$J_1 \neq 0$	$J_1 \neq 0$
$-1 < q < 0$ [Type II (“sudden”) singularity]	$H \to H_{\mathrm{s}}$	$\dot{H} \to \infty$		$J_2 \neq 0$
$q < -1$, but q is not any integer [Type IV singularity]	$H \to H_{\mathrm{s}}$	$\dot{H} \to 0$ (Higher derivatives of H diverge.)		

Gravitational field equations

$$-f + 2TF = 0$$: Consistency condition (Friedmann equation)

$$-F - 2TF' = 0 \longleftarrow \dot{H} \neq 0$$

- **Power-law model** $\boxed{f(T) = AT^{\alpha}}$ $A(\neq 0)$, $\alpha(\neq 0)$

$$\Rightarrow \quad F + 2TF' = A\,(-6)^{\alpha-1}\,(2\alpha-1)\,H^{2(\alpha-1)}\,\underline{=0}$$

$$-f + 2TF = A\,(-6)^{\alpha}\,(2\alpha-1)\,H^{2\alpha}\,\underline{=0}$$

< Removing the finite-time future singularities >

- **Power-law correction term**

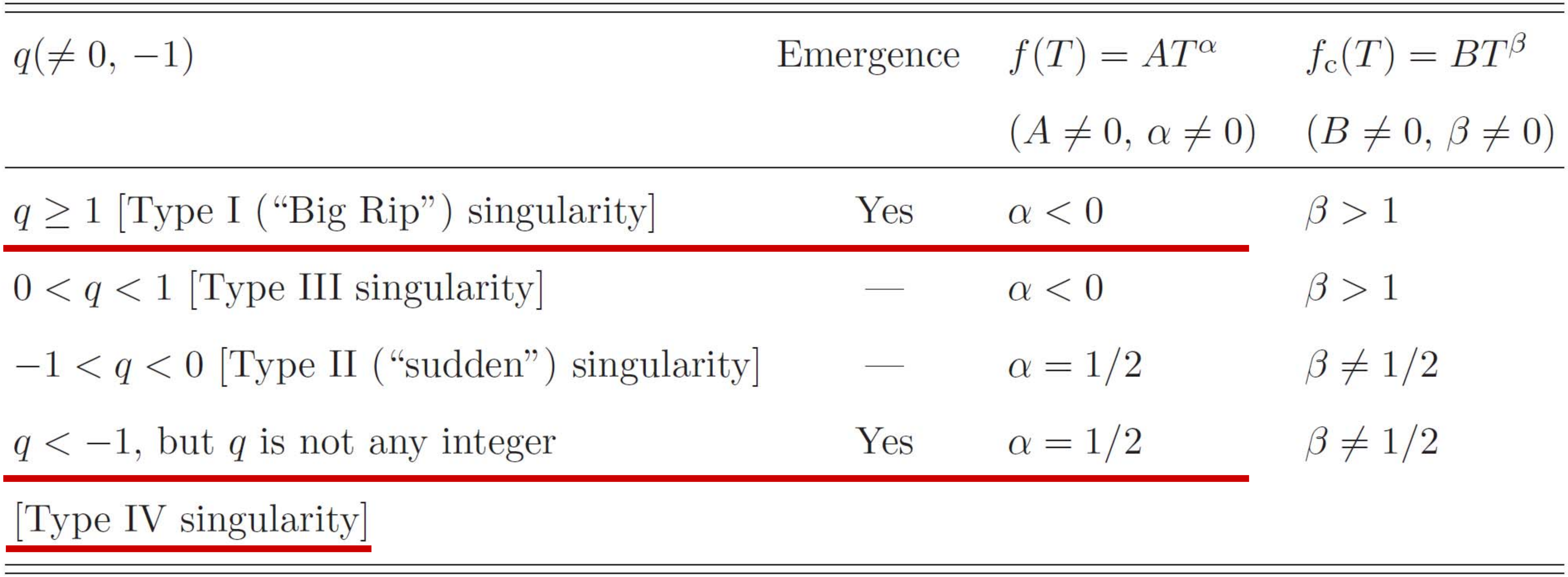
$$\boxed{f_{\mathrm{c}}(T) = BT^{\beta}} \quad \longrightarrow \quad f(T) = AT^{\alpha} + BT^{\beta}$$

$$\Rightarrow \quad -f + 2TF = A\left(2\alpha - 1\right)T^{\alpha} + B\left(2\beta - 1\right)T^{\beta} \underline{\neq 0}$$

$$-F - 2TF' = -A\alpha\left(2\alpha - 1\right)T^{\alpha-1} - B\beta\left(2\beta - 1\right)T^{\beta-1} \underline{\neq 0}$$

< Table 3 >

Necessary conditions for the appearance of the finite-time future singularities on a power-law $f(T)$ model and those for the removal of the finite-time future singularities on a power-law correction term $f_{\mathrm{c}}(T) = BT^{\beta}$.



$q(\neq 0,\ -1)$	Emergence	$f(T) = AT^{\alpha}$	$f_{\mathrm{c}}(T) = BT^{\beta}$
		$(A \neq 0,\ \alpha \neq 0)$	$(B \neq 0,\ \beta \neq 0)$
$q \geq 1$ [Type I ("Big Rip") singularity]	Yes	$\alpha < 0$	$\beta > 1$
$0 < q < 1$ [Type III singularity]	—	$\alpha < 0$	$\beta > 1$
$-1 < q < 0$ [Type II ("sudden") singularity]	—	$\alpha = 1/2$	$\beta \neq 1/2$
$q < -1$, but q is not any integer [Type IV singularity]	Yes	$\alpha = 1/2$	$\beta \neq 1/2$

< *f*(*T*) gravity models realizing cosmologies >

(a) (Power-law) Inflation

$$\boxed{H(t) \to \infty\,, \quad t \to 0}$$

$$H = \frac{h_{\mathrm{inf}}}{t}\,, \qquad h_{\mathrm{inf}} (> 1)$$

$$a(t) = a_{\mathrm{inf}} t^{h_{\mathrm{inf}}}$$

⇒

$$\boxed{\begin{array}{c} f(T) \;=\; AT^{\alpha} \\ \alpha < 0 \;\text{ or }\; \alpha = 1/2 \end{array}}$$

$$\underline{\ddot{a} = a_{\mathrm{inf}} h_{\mathrm{inf}} \left(h_{\mathrm{inf}} - 1\right) t^{h_{\mathrm{inf}} - 2} > 0}$$

← **Power-law inflation is realized.**

(b) Λ CDM model

$$\boxed{H(t) = H(t_0)\,, \quad t = t_0}$$

$$H^2 = \frac{\Lambda}{3}\,, \qquad \Lambda > 0$$

⇒

$$\boxed{f(T) = T - 2\Lambda}$$

$$\underline{a = a_{\Lambda} \exp\left(H_{\Lambda} t\right)\,, \quad a_{\Lambda} (> 0)}$$

← **Exponential expansion is realized.**

$$H \equiv H_{\Lambda} = \sqrt{\Lambda/3} = \mathrm{constant}\,, \;\; H_{\Lambda} (> 0)$$

$$q_{\rm dec} \equiv -\frac{1}{aH^2}\frac{d^2 a}{dt^2}$$: **Deceleration parameter**

$$j \equiv \frac{1}{aH^3}\frac{d^3 a}{dt^3}$$: **Jerk parameter**

$$s \equiv \frac{j-1}{3\left(q_{\rm dec}-1/2\right)}$$: **Snap parameter**

[Chiba and Nakamura, Prog. Theor. Phys. 100, 1077 (1998)]

[Sahni, Saini, Starobinsky and Alam, JETP Lett. 77, 201 (2003)
[Pisma Zh. Eksp. Teor. Fiz. 77, 249 (2003)]]

ΛCDM model

$$\boxed{\left(w_{\rm DE}, q_{\rm dec}, j, s\right) = \left(-1, -1, 1, 0\right)}$$

⇨ **These four parameter can be used to test models.**

(c) Little Rip cosmology

$$\boxed{H(t) \to \infty\,, \quad t \to \infty}$$

- **A scenario to avoid the Big Rip singularity.**

→ ρ_{DE} **increases in time with** $w_{\mathrm{DE}} < -1$ **and** w_{DE} **asymptotically approaches** $w_{\mathrm{DE}} = -1$ **.**

[Frampton, Ludwick and Scherrer, Phys. Rev. D 84, 063003 (2011)]

[Frampton, Ludwick, Nojiri, Odintsov and Scherrer, Phys. Lett. B 708, 204 (2012)]

$$H = H_{\mathrm{LR}} \exp\left(\xi t\right), \quad H_{\mathrm{LR}}(>\ 0), \ \xi(>\ 0)$$

$$a = a_{\mathrm{LR}} \exp\left[\frac{H_{\mathrm{LR}}}{\xi} \exp\left(\xi t\right)\right]$$

$$\Longrightarrow \quad \boxed{\begin{array}{l} f(T) \ = \ AT^{\alpha} \\ \alpha < 0 \ \text{ or } \ \alpha = 1/2 \end{array}}$$

$$w_{\mathrm{DE}} = -1 - \frac{2\xi}{3H_{\mathrm{LR}}} \exp\left(-\xi t\right) \longrightarrow w_{\mathrm{DE}} \ = \ -1 - 2H_0/\left(3H_{\mathrm{LR}}e\right)$$

$$\xi \ = \ H_0$$

$$t \ = \ t_0 \ \approx \ H_0^{-1}$$

→ Current values of the four parameters

$$w_{\rm DE}(t=t_0) = -1 - \frac{2}{3}\chi\,, \qquad q_{\rm dec}(t=t_0) = -1 - \chi$$

$$j(t=t_0) = 1 + \chi\,(\chi+3)\,, \qquad s(t=t_0) = -\frac{2\chi\,(\chi+3)}{3\,(2\chi+3)}$$

$$\boxed{\chi \equiv \frac{H_0}{H_{\rm LR}e} \leq 0.36}$$

Λ CDM model

$$\boxed{(w_{\rm DE}, q_{\rm dec}, j, s) = (-1, -1, 1, 0)}$$

⇨ **If we take $\chi \ll 1$, this Little Rip model can be compatible with the Λ CDM model.**

(d) Pseudo Rip cosmology

$$\boxed{H(t) \to H_\infty < \infty\,, \quad t \to \infty}$$

- **A phantom scenario with the universe approaching $H_\infty (> 0)$ de Sitter phase.**

→ $H(t)$ **approaches to a finite value in the limit $t \to \infty$.**
This behavior is different from Little Rip cosmology.

[Frampton, Ludwick and Scherrer, Phys. Rev. D 85, 083001 (2012)]

[Astashenok, Nojiri, Odintsov and Yurov, Phys. Lett. B 709, 396 (2012)]

$$H(t) = H_{\mathrm{PR}} \tanh\left(\frac{t}{t_0}\right), \quad H_{\mathrm{PR}} (> 0)\ , \quad t \geq t_0$$

$$a = a_{\mathrm{PR}} \cosh\left(\frac{t}{t_0}\right)$$

$$w_{\mathrm{DE}} = -1 - \frac{2}{3t_0 H_{\mathrm{PR}}} \frac{1}{\sinh^2 (t/t_0)}$$

⇨

$$\boxed{f(T) = A\sqrt{T}}$$

→ Current values of the four parameters

$$w_{\mathrm{DE}}(t=t_0) = -1 - \frac{2\delta}{3\sinh^2 1}\,, \qquad q_{\mathrm{dec}}(t=t_0) = -1 + \frac{\delta^2\tanh^2 1 - 1}{\delta^2\tanh^2 1}$$

$$j(t=t_0) = 1 + \frac{1-\delta^3\tanh^2 1}{\delta^3\tanh^2 1}\,, \qquad s(t=t_0) = \frac{2}{3\delta}\frac{\delta^3\tanh^2 1 - 1}{\delta^2\tanh^2 1 + 2}$$

$$\boxed{\delta \equiv \frac{H_0}{H_{\mathrm{PR}}} \leq 0.497196}$$

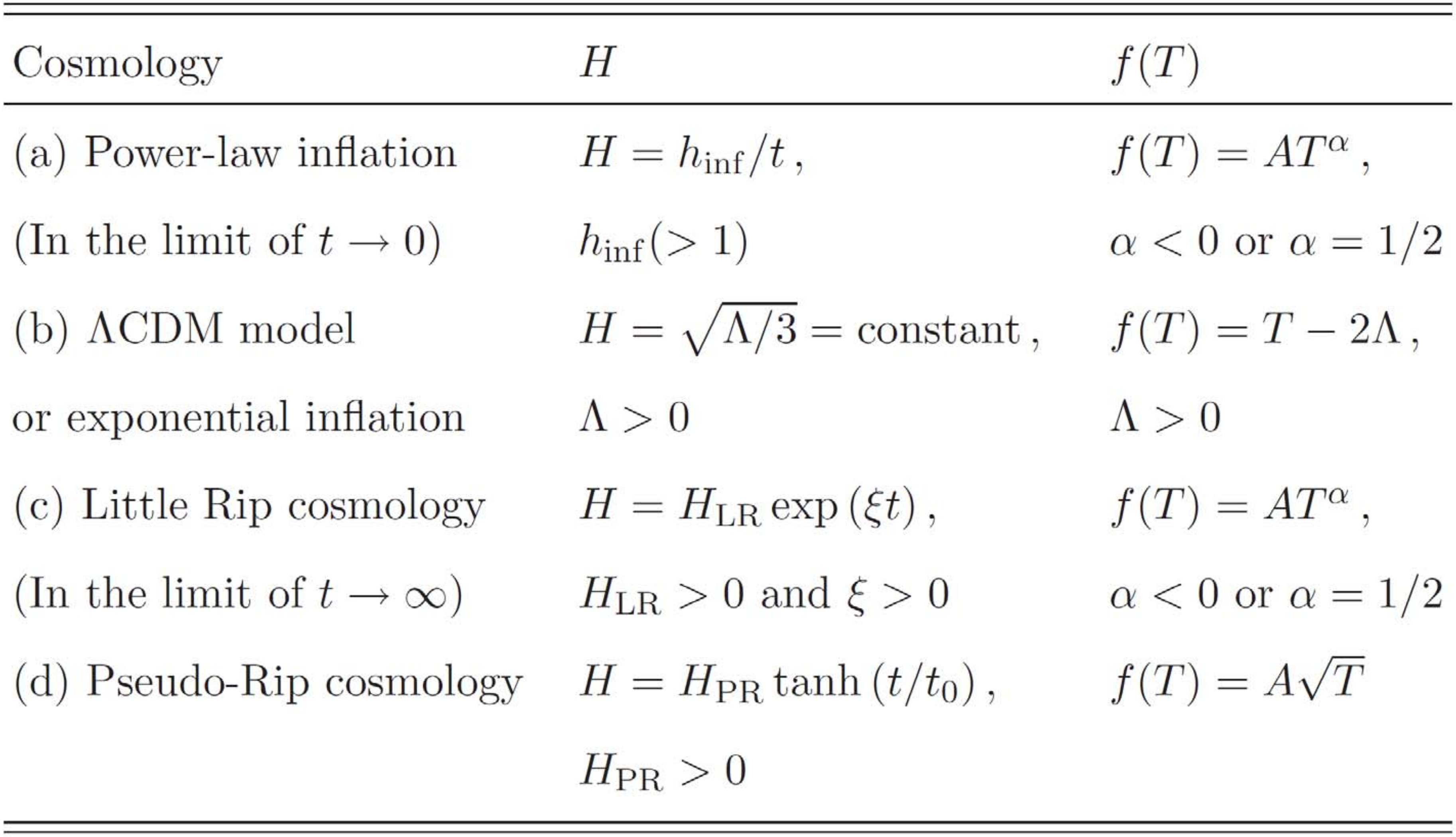
Λ CDM model

$$\boxed{(w_{\mathrm{DE}}, q_{\mathrm{dec}}, j, s) = (-1, -1, 1, 0)}$$

We can take an appropriate value of δ so that this Pseudo-Rip model can be consistent with the Λ CDM model.

< Table 4 >

Forms of H and $f(T)$ with realizing (a) inflation in the early universe, (b) the Λ CDM model, (c) Little Rip cosmology and (d) Pseudo-Rip cosmology.



Cosmology	H	$f(T)$
(a) Power-law inflation	$H = h_{\mathrm{inf}}/t\,,$	$f(T) = AT^{\alpha}\,,$
(In the limit of $t \to 0$)	$h_{\mathrm{inf}} (> 1)$	$\alpha < 0$ or $\alpha = 1/2$
(b) ΛCDM model	$H = \sqrt{\Lambda/3} = \mathrm{constant}\,,$	$f(T) = T - 2\Lambda\,,$
or exponential inflation	$\Lambda > 0$	$\Lambda > 0$
(c) Little Rip cosmology	$H = H_{\mathrm{LR}} \exp(\xi t)\,,$	$f(T) = AT^{\alpha}\,,$
(In the limit of $t \to \infty$)	$H_{\mathrm{LR}} > 0$ and $\xi > 0$	$\alpha < 0$ or $\alpha = 1/2$
(d) Pseudo-Rip cosmology	$H = H_{\mathrm{PR}} \tanh(t/t_0)\,,$	$f(T) = A\sqrt{T}$
	$H_{\mathrm{PR}} > 0$	

Inertial force on a particle with mass m in the expanding universe

$$a_0 \equiv a(t = t_0) = 1$$

$$F_{\mathrm{inert}} = ml\frac{\ddot{a}}{a} = ml\left(\dot{H} + H^2\right)$$

$$= -ml\frac{\kappa^2}{6}\left(\rho_{\mathrm{DE}}(a) + 3P_{\mathrm{DE}}(a)\right) = ml\frac{\kappa^2}{6}\left(2\rho_{\mathrm{DE}}(a) + \frac{d\rho_{\mathrm{DE}}(a)}{da}a\right)$$

[Frampton, Ludwick and Scherrer, Phys. Rev. D 84, 063003 (2011)]

[Frampton, Ludwick, Nojiri, Odintsov and Scherrer, Phys. Lett. B 708, 204 (2012)]

- **We provide that two particles are bound by a constant force F_{b} .**

When $F_{\mathrm{inert}} > F_{\mathrm{b}}$, the two particles become unbound and hence the bound structure is dissociated.

$F_{\mathrm{inert}}\ (> 0)$

For a Big Rip singularity

$$F_{\mathrm{inert}} = mlh_{\mathrm{s}}\left[\frac{q}{(t_{\mathrm{s}} - t)^{q+1}} + \frac{h_{\mathrm{s}}}{(t_{\mathrm{s}} - t)^{-2q}}\right] \longrightarrow \underline{\infty\,,} \quad t \to t_{\mathrm{s}}$$

For Little Rip cosmology

$$F_{\mathrm{inert}} = mlH_{\mathrm{LR}}\left[\xi + H_{\mathrm{LR}}\exp\left(\xi t\right)\right]\exp\left(\xi t\right) \longrightarrow \underline{\infty\,,} \quad t \to \infty$$

For Pseudo Rip cosmology

$$F_{\mathrm{inert}} = mlH_{\mathrm{PR}} \left[\frac{1}{t_0 \cosh^2 (t/t_0)} + H_{\mathrm{PR}} \tanh^2 \left(\frac{t}{t_0} \right) \right] \longrightarrow \underline{F_{\mathrm{inert}\,,\infty}^{\mathrm{PR}} < \infty\,,} \quad t \to \infty$$

$\uparrow$

F_{inert} **asymptotically approaches to a finite value.**

< Earth-Sun (ES) system >

$$F_{\mathrm{inert}\,,\infty}^{\mathrm{PR}} \equiv mlH_{\mathrm{PR}}^2$$

$$F_{\mathrm{b}}^{\mathrm{ES}} = GM_{\oplus}M_{\odot}/r_{\oplus-\odot}^2 = 4.37 \times 10^{16}\,\mathrm{GeV}^2$$

$$r_{\oplus-\odot} = 1\mathrm{AU} = 7.5812 \times 10^{26}\,\mathrm{GeV}^{-1}\,, \quad M_{\oplus} = 3.357 \times 10^{51}\,\mathrm{GeV}$$

$$M_{\odot} = 1.116 \times 10^{57}\,\mathrm{GeV}$$

Condition for the disintegration of the ES system

$$\boxed{F_{\mathrm{inert}\,,\infty}^{\mathrm{PR}} > F_{\mathrm{b}}^{\mathrm{ES}}} \longrightarrow H_{\mathrm{PR}} > \sqrt{GM_{\odot}/r_{\oplus-\odot}^3} = 1.31 \times 10^{-31}\,\mathrm{GeV}$$

If this condition is met, the disintegration of the ES system can occur much before arriving at de Sitter universe, so that the Pseudo-Rip scenario can be realized.

IV. Summary

- **We have discussed modified gravitational theories to explain the current accelerated expansion of the universe, so-called dark energy problem.**
- **We have explicitly shown that three types of the finite-time future singularities (Type I, II and III) can occur in non-local gravity and examined their properties.**
- **We have illustrated that there appear finite-time future singularities (Type I and IV) in $f(T)$ gravity and reconstructed an $f(T)$ gravity model with realizing the finite-time future singularities.**

⇨ **We have verified that a power-law type correction term T^β ($\beta > 1$) such as a T^2 term can remove the finite-time future singularities in $f(T)$ gravity. This is the same feature as in $F(R)$ gravity.**

- **We have derived the expressions of *f*(*T*) gravity models in which (a) (Power-law) Inflation, (b) Λ CDM model, (c) Little Rip cosmology, and (d) Pseudo Rip Cosmology can be realized.**

(a) (Power-law) Inflation

$$\boxed{H(t) \to \infty\,, \quad t \to 0}$$

(b) Λ CDM model

$$\boxed{H(t) = H(t_0)\,, \quad t = t_0}$$

(c) Little Rip cosmology

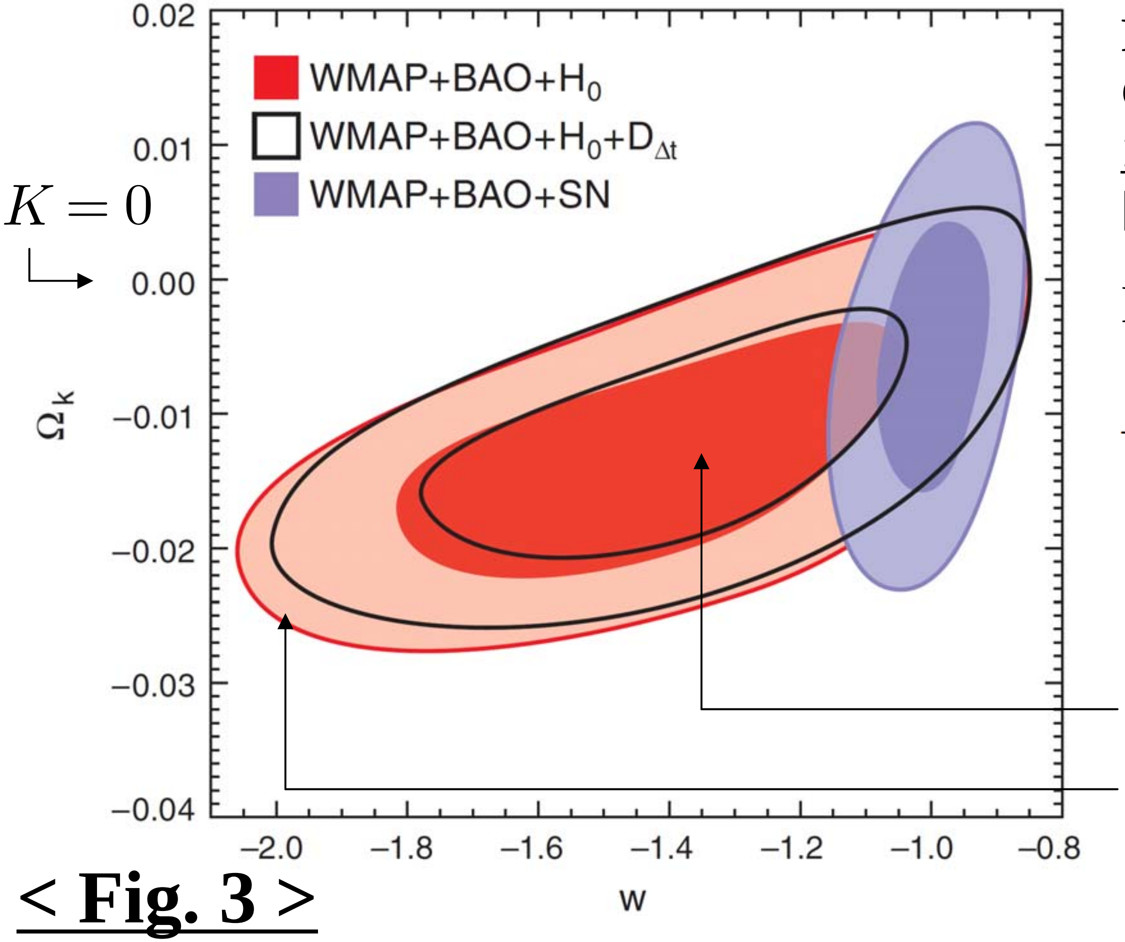
$$\boxed{H(t) \to \infty\,, \quad t \to \infty}$$

(d) Pseudo Rip cosmology

$$\boxed{H(t) \to H_\infty < \infty\,, \quad t \to \infty}$$

Backup Slides

< 7-year WMAP data on the current value of w >



(68% CL)

(95% CL)

From [Komatsu *et al*. [WMAP Collaboration], Astrophys. J. Suppl. 192, 18 (2011) [arXiv:1001.4538 [astro-ph.CO]]].

Hubble constant (H_0) measurement

$D_{\Delta t}$: Time delay distance

$\Omega_K \equiv \frac{K}{(a_0 H_0)^2}$: Density parameter for the curvature

$K = 0$: Flat universe

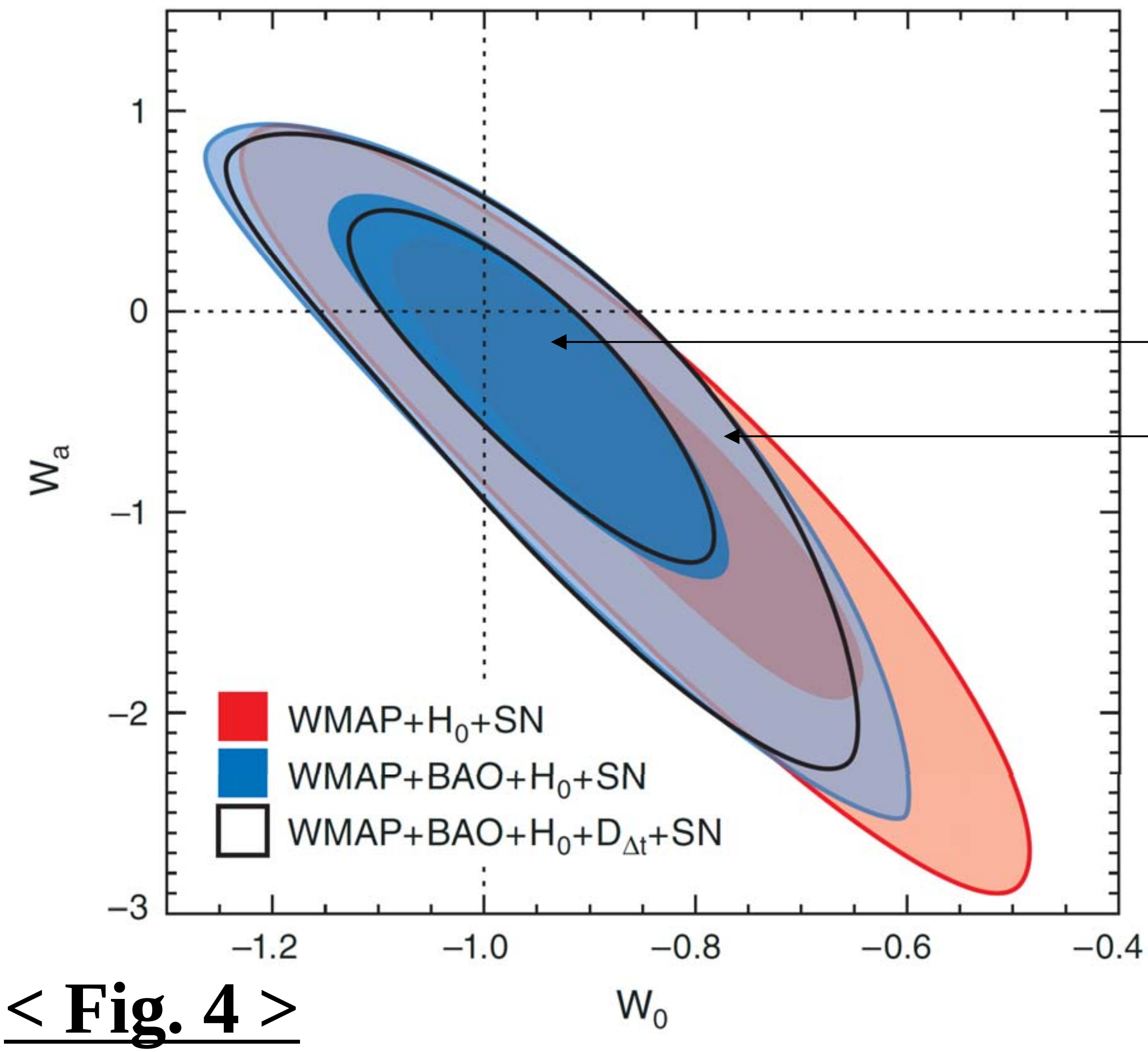
< Fig. 3 >

- For the flat universe, constant w :

$$\boxed{w = -1.10 \pm 0.14 \ (68\% \text{ CL})}$$

(From *WMAP* +BAO+H_0 .)

Cf. $\Omega_\Lambda = 0.725 \pm 0.016$ (68% CL)



From [E. Komatsu *et al*. [WMAP Collaboration], Astrophys. J. Suppl. 192, 18 (2011) [arXiv:1001.4538 [astro-ph.CO]]].

(68% CL)

(95% CL)

Time-dependent w

$$w(a) = w_0 + w_a(1 - a)$$

$$a = \frac{1}{1+z}$$

w_0: Current value of w

z : Redshift

(From $WMAP$+BAO +H_0+SN.)

< Fig. 4 >

- For the flat universe, a variable EoS :

$$w_0 = -0.93 \pm 0.13\,,\ w_a = -0.41^{+0.72}_{-0.71}\ \ (68\%\ \mathrm{CL})$$

- $$\boxed{\eta = -\int^{t} \frac{1}{a^3} \left(\int^{\bar{t}} R a^3 d\bar{t} \right) dt}$$

η_{c} : Integration constant

- We take a form of $f(\eta)$ as $\boxed{f(\eta) = f_{\mathrm{s}} \eta^{\sigma}}$.

$f_{\mathrm{s}}(\neq 0)$, $\sigma(\neq 0)$

: Non-zero constants

- $$\boxed{\xi = \int^{t} \frac{1}{a^3} \left(\int^{\bar{t}} \frac{df(\eta)}{d\eta} R a^3 d\bar{t} \right) dt}$$

ξ_{c} : Integration constant

$\longrightarrow$ We examine the behavior of each term of the gravitational field equations in the limit $t \longrightarrow t_{\mathrm{s}}$, in particular that of the leading terms, and study the condition that an asymptotic solution can be obtained.

- For case (ii) $[q > 1,\ \sigma < 0]$, $\boxed{\xi_{\mathrm{c}} = 1}$
- For case (iii) $[-1 < q < 0\,,\ 0 < q < 1]$,

$$\boxed{f_{\mathrm{s}} \eta_{\mathrm{c}}^{\sigma - 1} \left(6\sigma - \eta_{\mathrm{c}} \right) + \xi_{\mathrm{c}} - 1 = 0}$$

⇒ the leading term vanishes in both gravitational field equations.

$$\boxed{H \sim \frac{h_{\mathrm{s}}}{\left(t_{\mathrm{s}} - t \right)^{q}}}$$

⇒ Thus, the expression of the Hubble parameter can be a leading-order solution in terms of $(t_{\mathrm{s}} - t)$ for the gravitational field.

⇓

This implies that there can exist the finite-time future singularities in non-local gravity.

(c) Little Rip cosmology

$$\boxed{H(t) \to \infty \,, \quad t \to \infty}$$

→ **A scenario to avoid the Big Rip singularity.**
ρ_{DE} **increases in time with** $w_{\mathrm{DE}} < -1$ **and** w_{DE} **asymptotically approaches** $w_{\mathrm{DE}} = -1$ **.**

[Frampton, Ludwick and Scherrer, Phys. Rev. D 84, 063003 (2011)]

[Frampton, Ludwick, Nojiri, Odintsov and Scherrer, Phys. Lett. B 708, 204 (2012)]

$$H = H_{\mathrm{LR}} \exp(\xi t), \quad H_{\mathrm{LR}}(> \ 0), \quad \xi(> \ 0)$$

$$a = a_{\mathrm{LR}} \exp\left[\frac{H_{\mathrm{LR}}}{\xi} \exp(\xi t)\right]$$

$$\Rightarrow$$

$$\boxed{\begin{array}{c} f(T) \ = \ A T^{\alpha} \\ \alpha < 0 \ \text{ or } \ \alpha = 1/2 \end{array}}$$

$$w_{\mathrm{DE}} = -1 - \frac{2\xi}{3H_{\mathrm{LR}}} \exp(-\xi t) \longrightarrow w_{\mathrm{DE}} \ = \ -1 - 2H_0/\left(3H_{\mathrm{LR}} e\right)$$

$$\xi \ = \ H_0 \,, \quad t \ = \ t_0 \approx H_0^{-1}$$

$$w_{\mathrm{DE}} = \ -1.084 \pm 0.063 \ \ (68\% \ \mathrm{CL})$$

[Hinshaw *et al.*, arXiv:1212.5226 [astro-ph.CO]]

$H_0 = 2.1h \times 10^{-42}\,\mathrm{GeV}$: Current value of H, $h = 0.7$

[Freedman *et al.* [HST Collaboration], Astrophys. J. 553, 47 (2001)]

→ $H_{\mathrm{LR}} \geq [2H_0/(3e)]\,(1/0.24) = 1.50 \times 10^{-42}\,\mathrm{GeV}$

$$q_{\mathrm{dec}} = -1 - \frac{\xi}{H_{\mathrm{LR}} \exp(\xi t)}$$

$$j = 1 + \frac{\xi}{H_{\mathrm{LR}}} \left[\frac{\xi}{H_{\mathrm{LR}} \exp(\xi t)} + 3 \right] \frac{1}{\exp(\xi t)}$$

$$s = -\frac{2\xi \left[\xi + 3H_{\mathrm{LR}} \exp(\xi t)\right]}{3H_{\mathrm{LR}} \left[2\xi + 3H_{\mathrm{LR}} \exp(\xi t)\right] \exp(\xi t)}$$

Λ CDM model

$$\boxed{\begin{aligned}&(w_{\mathrm{DE}}, q_{\mathrm{dec}}, j, s)\\ &= \; (-1, -1, 1, 0)\end{aligned}}$$

→ **Current values of the four parameters**

$$w_{\mathrm{DE}}(t = t_0) = -1 - \frac{2}{3}\chi\,, \qquad q_{\mathrm{dec}}(t = t_0) = -1 - \chi$$

$$j(t = t_0) = 1 + \chi\,(\chi + 3)\,, \qquad s(t = t_0) = -\frac{2\chi\,(\chi + 3)}{3\,(2\chi + 3)}$$

$$\chi \equiv \frac{H_0}{H_{\mathrm{LR}} e} \leq 0.36$$

⇨ **If we take $\chi \ll 1$ enough for the deviation of the values of the four parameters $(w_{\mathrm{DE}}, q_{\mathrm{dec}}, j, s)$ from those for the Λ CDM model $(-1, -1, 1, 0)$ to be very small, this Little Rip model can be compatible with the Λ CDM model.**

$$q_{\mathrm{dec}} = -1 + \frac{\left(t_0 H_{\mathrm{PR}}\right)^2 \tanh^2\left(t/t_0\right) - 1}{\left(t_0 H_{\mathrm{PR}}\right)^2 \tanh^2\left(t/t_0\right)}\,, \quad j = 1 + \frac{1 - \left(t_0 H_{\mathrm{PR}}\right)^3 \tanh^2\left(t/t_0\right)}{\left(t_0 H_{\mathrm{PR}}\right)^3 \tanh^2\left(t/t_0\right)}$$

$$s = \frac{2}{3 t_0 H_{\mathrm{PR}}} \frac{\left(t_0 H_{\mathrm{PR}}\right)^3 \tanh^2\left(t/t_0\right) - 1}{\left(t_0 H_{\mathrm{PR}}\right)^2 \tanh^2\left(t/t_0\right) + 2}$$

Λ CDM model

$$\boxed{\left(w_{\mathrm{DE}}, q_{\mathrm{dec}}, j, s\right) = (-1, -1, 1, 0)}$$

→ **Current values of the four parameters**

$$w_{\mathrm{DE}}(t = t_0) = -1 - \frac{2\delta}{3 \sinh^2 1}\,, \qquad q_{\mathrm{dec}}(t = t_0) = -1 + \frac{\delta^2 \tanh^2 1 - 1}{\delta^2 \tanh^2 1}$$

$$j(t = t_0) = 1 + \frac{1 - \delta^3 \tanh^2 1}{\delta^3 \tanh^2 1}\,, \quad s(t = t_0) = \frac{2}{3\delta} \frac{\delta^3 \tanh^2 1 - 1}{\delta^2 \tanh^2 1 + 2}$$

$$\delta \equiv \frac{H_0}{H_{\mathrm{PR}}} \leq 0.497196\,, \qquad \delta \leq (3/2)\, 0.24 \sinh^2 1 = 0.497196$$

⇨ **We can take an appropriate value of δ in order for the deviation of the values of the four parameters $\left(w_{\mathrm{DE}}, q_{\mathrm{dec}}, j, s\right)$ from those for the Λ CDM model $(-1, -1, 1, 0)$ to be very small, so that this Pseudo-Rip model can be consistent with the Λ CDM model.**

(d) Pseudo Rip cosmology

$$\boxed{H(t) \to H_{\infty} < \infty \,, \quad t \to \infty}$$

→ **A phantom scenario with the universe approaching $H_{\infty} (> 0)$ de Sitter phase.**

$H(t)$ approaches to a finite value in the limit $t \to \infty$.

This behavior is different from Little Rip cosmology.

[Frampton, Ludwick and Scherrer, Phys. Rev. D 85, 083001 (2012)]

[Astashenok,, Nojiri, Odintsov and Yurov, Phys. Lett. B 709, 396 (2012)]

$$H(t) = H_{\mathrm{PR}} \tanh \left(\frac{t}{t_0} \right), \quad H_{\mathrm{PR}} (> 0) \,, \quad t \geq t_0$$

$$a = a_{\mathrm{PR}} \cosh \left(\frac{t}{t_0} \right), \quad w_{\mathrm{DE}} = -1 - \frac{2}{3 t_0 H_{\mathrm{PR}}} \frac{1}{\sinh^2 (t/t_0)} \,,$$

$$\Longrightarrow \quad \boxed{f(T) = A\sqrt{T}}$$

$$w_{\mathrm{DE}} = \; -1.084 \pm 0.063 \;\; (68\% \text{ CL})$$

[Hinshaw *et al.*, arXiv:1212.5226 [astro-ph.CO]]

$$H_0 = 2.1 h \times 10^{-42} \, \mathrm{GeV} \,, \quad h = 0.7$$

[Freedman *et al.* [HST Collaboration], Astrophys. J. 553, 47 (2001)]

$$\longrightarrow \quad H_{\mathrm{PR}} \;\; \geq \; (2H_0/3) \left[4/ \left(e - e^{-1} \right)^2 \right] (1/0.24) = 2.96 \times 10^{-42} \, \mathrm{GeV}$$

$$q_{\mathrm{dec}} = -1 + \frac{\left(t_0 H_{\mathrm{PR}}\right)^2 \tanh^2\left(t/t_0\right) - 1}{\left(t_0 H_{\mathrm{PR}}\right)^2 \tanh^2\left(t/t_0\right)}\,, \qquad j = 1 + \frac{1 - \left(t_0 H_{\mathrm{PR}}\right)^3 \tanh^2\left(t/t_0\right)}{\left(t_0 H_{\mathrm{PR}}\right)^3 \tanh^2\left(t/t_0\right)}$$

$$s = \frac{2}{3 t_0 H_{\mathrm{PR}}} \frac{\left(t_0 H_{\mathrm{PR}}\right)^3 \tanh^2\left(t/t_0\right) - 1}{\left(t_0 H_{\mathrm{PR}}\right)^2 \tanh^2\left(t/t_0\right) + 2}$$

Λ CDM model

$$\boxed{\left(w_{\mathrm{DE}}, q_{\mathrm{dec}}, j, s\right) = \left(-1, -1, 1, 0\right)}$$

→ Current values of the four parameters

$$w_{\mathrm{DE}}(t = t_0) = -1 - \frac{2\delta}{3 \sinh^2 1}\,, \qquad q_{\mathrm{dec}}(t = t_0) = -1 + \frac{\delta^2 \tanh^2 1 - 1}{\delta^2 \tanh^2 1}$$

$$j(t = t_0) = 1 + \frac{1 - \delta^3 \tanh^2 1}{\delta^3 \tanh^2 1}\,, \qquad s(t = t_0) = \frac{2}{3\delta} \frac{\delta^3 \tanh^2 1 - 1}{\delta^2 \tanh^2 1 + 2}$$

$$\boxed{\delta \equiv \frac{H_0}{H_{\mathrm{PR}}} \leq 0.497196} \longleftarrow \delta \leq (3/2)\, 0.24 \sinh^2 1 = 0.497196$$

⇨ **We can take an appropriate value of δ so that this Pseudo-Rip model can be consistent with the Λ CDM model.**

→ For Pseudo Rip cosmology

$$F_{\mathrm{inert}} = mlH_{\mathrm{PR}}\left[\frac{1}{t_0\cosh^2(t/t_0)} + H_{\mathrm{PR}}\tanh^2\left(\frac{t}{t_0}\right)\right] \longrightarrow \underline{F_{\mathrm{inert},\infty}^{\mathrm{PR}} < \infty}\,, \quad t\to\infty$$

$$F_{\mathrm{inert},\infty}^{\mathrm{PR}} \equiv mlH_{\mathrm{PR}}^2$$

↑ F_{inert} **asymptotically approaches to a finite value.**

< Earth-Sun (ES) system >

$$F_{\mathrm{b}}^{\mathrm{ES}} = GM_{\oplus}M_{\odot}/r_{\oplus-\odot}^2 = 4.37\times10^{16}\,\mathrm{GeV}^2 \qquad m = M_{\oplus}$$

$$r_{\oplus-\odot} = 1\mathrm{AU} = 7.5812\times10^{26}\,\mathrm{GeV}^{-1} \qquad l = r_{\oplus-\odot}$$

$$M_{\oplus} = 3.357\times10^{51}\,\mathrm{GeV}\,, \quad M_{\odot} = 1.116\times10^{57}\,\mathrm{GeV}$$

Condition for the disintegration of the ES system

$$\boxed{F_{\mathrm{inert},\infty}^{\mathrm{PR}} > F_{\mathrm{b}}^{\mathrm{ES}}}$$

$$\longrightarrow \quad H_{\mathrm{PR}} > \sqrt{GM_{\odot}/r_{\oplus-\odot}^3} = 1.31\times10^{-31}\,\mathrm{GeV}$$

If this condition is met, the disintegration of the ES system can occur much before arriving at de Sitter universe, so that the Pseudo-Rip scenario can be realized.

* **The constraint from the current value of w_{DE} is much weaker as** $H_{\mathrm{PR}} \geq 2.96\times10^{-42}\,\mathrm{GeV}$

< Canonical scalar field >

$$S_\phi = \int d^4x \sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]$$

$g = \det(g_{\mu\nu})$

ϕ : Scalar field

$V(\phi)$: Potential of ϕ

- For a homogeneous scalar field $\phi = \phi(t)$:

$\longrightarrow$ $\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)$

$$\Rightarrow \boxed{w_\phi = \frac{P_\phi}{\rho_\phi} = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)}}$$

If $\dot{\phi}^2 \ll V(\phi)$, $\underline{w_\phi \approx -1}$.

$\longrightarrow$ **Accelerated expansion can be realized.**

< *F*(*R*) gravity >

$$S = \int d^4x \sqrt{-g}\, \underline{\frac{F(R)}{2\kappa^2}}$$

$$\boxed{F(R)\ \text{gravity}}$$

$F(R) = R$

: General Relativity

[Nojiri and Odintsov, Phys. Rept. 505, 59 (2011) [arXiv:1011.0544 [gr-qc]]; Int. J. Geom. Meth. Mod. Phys. 4, 115 (2007) [arXiv:hep-th/0601213]]

[Capozziello and Francaviglia, Gen. Rel. Grav. 40, 357 (2008)]

[Sotiriou and Faraoni, Rev. Mod. Phys. 82, 451 (2010)]

[De Felice and Tsujikawa, Living Rev. Rel. 13, 3 (2010)]

[Capozziello and Faraoni, *Beyond Einstein Gravity* (Springer, 2010)]

[Capozziello and De Laurentis, Phys. Rept. 509, 167 (2011)]

[Clifton, Ferreira, Padilla and Skordis, Phys. Rept. 513, 1 (2012)]

< Gravitational field equation >

$$F'(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}F(R) + g_{\mu\nu}\Box F'(R) - \nabla_\mu\nabla_\nu F'(R) = 0$$

$F'(R) = dF(R)/dR$

$\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$: Covariant d'Alembertian

∇_μ : Covariant derivative operator

- In the flat FLRW background, gravitational field equations read

$$H^2 = \frac{\kappa^2}{3}\rho_{\mathrm{eff}}\,, \quad \dot{H} = -\frac{\kappa^2}{2}\left(\rho_{\mathrm{eff}} + P_{\mathrm{eff}}\right)$$

$\rho_{\mathrm{eff}}, \; P_{\mathrm{eff}}$: Effective energy density and pressure from the term $F(R) - R$

$$\rho_{\mathrm{eff}} = \frac{1}{\kappa^2 F'(R)}\left[\frac{1}{2}\left(-F(R) + RF'(R)\right) - 3H\dot{R}F''(R)\right]$$

$$P_{\mathrm{eff}} = \frac{1}{\kappa^2 F'(R)}\left[\frac{1}{2}\left(F(R) - RF'(R)\right) + \left(2H\dot{R} + \ddot{R}\right)F''(R) + \dot{R}^2 F'''(R)\right]$$

$$\longrightarrow \boxed{w_{\mathrm{eff}} = \frac{P_{\mathrm{eff}}}{\rho_{\mathrm{eff}}} = \frac{\left(F(R) - RF'(R)\right)/2 + \left(2H\dot{R} + \ddot{R}\right)F''(R) + \dot{R}^2 F'''(R)}{\left(-F(R) + RF'(R)\right)/2 - 3H\dot{R}F''(R)}}$$

- Example : $F(R) \propto R^n \; (n \neq 1)$

$$\longrightarrow a \propto t^q, \quad q = \frac{-2n^2+3n-1}{n-2}$$

$$w_{\mathrm{eff}} = -\frac{6n^2-7n-1}{6n^2-9n+3}$$

[Capozziello, Carloni and Troisi, Recent Res. Dev. Astron. Astrophys. 1, 625 (2003)]

If $q > 1$, accelerated expansion can be realized.

(For $n = 3/2$ or $n = -1$, $q = 2$ and $w_{\mathrm{eff}} = -2/3$.)

< Conditions for the viability of $F(R)$ gravity >

(1) $F'(R) > 0$ ← **Positivity of the effective gravitational coupling**

$F'(R) \equiv df(R)/dR$ $\quad G_{\mathrm{eff}} = G/F'(R) > 0$ $\quad G$: Gravitational constant

(2) $F''(R) > 0$ ← **Stability condition:** $M^2 \approx 1/(3F''(R)) > 0$

$F''(R) \equiv d^2F(R)/dR^2$

[Dolgov and Kawasaki, Phys. Lett. B 573, 1 (2003)]

M : Mass of a new scalar degree of freedom (“scalaron”) in the weak-field regime.

(3) $F(R) \to R - 2\Lambda$ for $R \gg R_0$. ← **Existence of a matter-dominated stage**

R_0 : Current curvature, Λ : Cosmological constant

(4) $0 < m \equiv RF''(R)/F'(R) < 1$ ← **Stability of the late-time de Sitter point**

[Amendola, Gannouji, Polarski and Tsujikawa, Phys. Rev. D 75, 083504 (2007)]

Cf. For general relativity, $m = 0$.

[Amendola and Tsujikawa, Phys. Lett. B 660, 125 (2008)]

[Faraoni and Nadeau, Phys. Rev. D 75, 023501 (2007)]

(5) Constraints from the violation of the equivalence principle

$M = M(R)$ ← Scale-dependence : **“Chameleon mechanism”**

Cf. [Khoury and Weltman, Phys. Rev. D 69, 044026 (2004)]

(6) Solar-system constraints

[Chiba, Phys. Lett. B 575, 1 (2003)]

[Chiba, Smith and Erickcek, Phys. Rev. D 75, 124014 (2007)]

< Models of *F*(*R*) gravity (examples) >

(i) Hu-Sawicki model

[Hu and Sawicki, Phys. Rev. D 76, 064004 (2007)]

Cf. [Nojiri and Odintsov, Phys. Lett. B 657, 238 (2007); Phys. Rev. D 77, 026007 (2008)]

$$F_{\mathrm{HS}} = R - \frac{c_1 R_{\mathrm{HS}} \left(R/R_{\mathrm{HS}}\right)^p}{c_2 \left(R/R_{\mathrm{HS}}\right)^p + 1}$$

$c_1,\ c_2,\ p(>0),\ R_{\mathrm{HS}}(>0)$

: Constant parameters

(ii) Starobinsky's model

[Starobinsky, JETP Lett. 86, 157 (2007)]

$$F_{\mathrm{S}} = R + \lambda R_{\mathrm{S}} \left[\left(1 + \frac{R^2}{R_{\mathrm{S}}^2} \right)^{-n} - 1 \right]$$

$\lambda(>0),\ n(>0),\ R_{\mathrm{S}}$

: Constant parameters

(iii) Hyperbolic model

[Tsujikawa, Phys. Rev. D 77, 023507 (2008)]

$$F_{\mathrm{H}} = R - \mu R_{\mathrm{H}} \tanh\left(\frac{R}{R_{\mathrm{H}}}\right)$$

$\mu(>0),\ R_{\mathrm{H}}(>0)$

: Constant parameters

(iv) Exponential gravity model

[Cognola, Elizalde, Nojiri, Odintsov, Sebastiani and Zerbini, Phys. Rev. D 77, 046009 (2008)]

[Linder, Phys. Rev. D 80, 123528 (2009)]

$$F_{\mathrm{E}} = R - \beta R_{\mathrm{E}} \left(1 - \mathrm{e}^{-R/R_{\mathrm{E}}}\right)$$

$\beta,\ R_{\mathrm{E}}$: Constant parameters

(v) Appleby-Battye model

[Appleby and Battye, Phys. Lett. B 654, 7 (2007)]

$$F_{\mathrm{AB}}(R) = \frac{R}{2} + \frac{1}{2b_1}\log\left[\cosh(b_1 R) - \tanh(b_2)\sinh(b_1 R)\right]$$

$b_1(> 0),\ \ b_2$

: Constant parameters

(vi) Power-law model

[Amendola, Gannouji, Polarski and Tsujikawa, Phys. Rev. D 75, 083504 (2007)]

[Li and Barrow, Phys. Rev. D 75, 084010 (2007)]

$$F(R) = R - \mu R^{v}$$

$\mu(> 0)$: Constant parameter

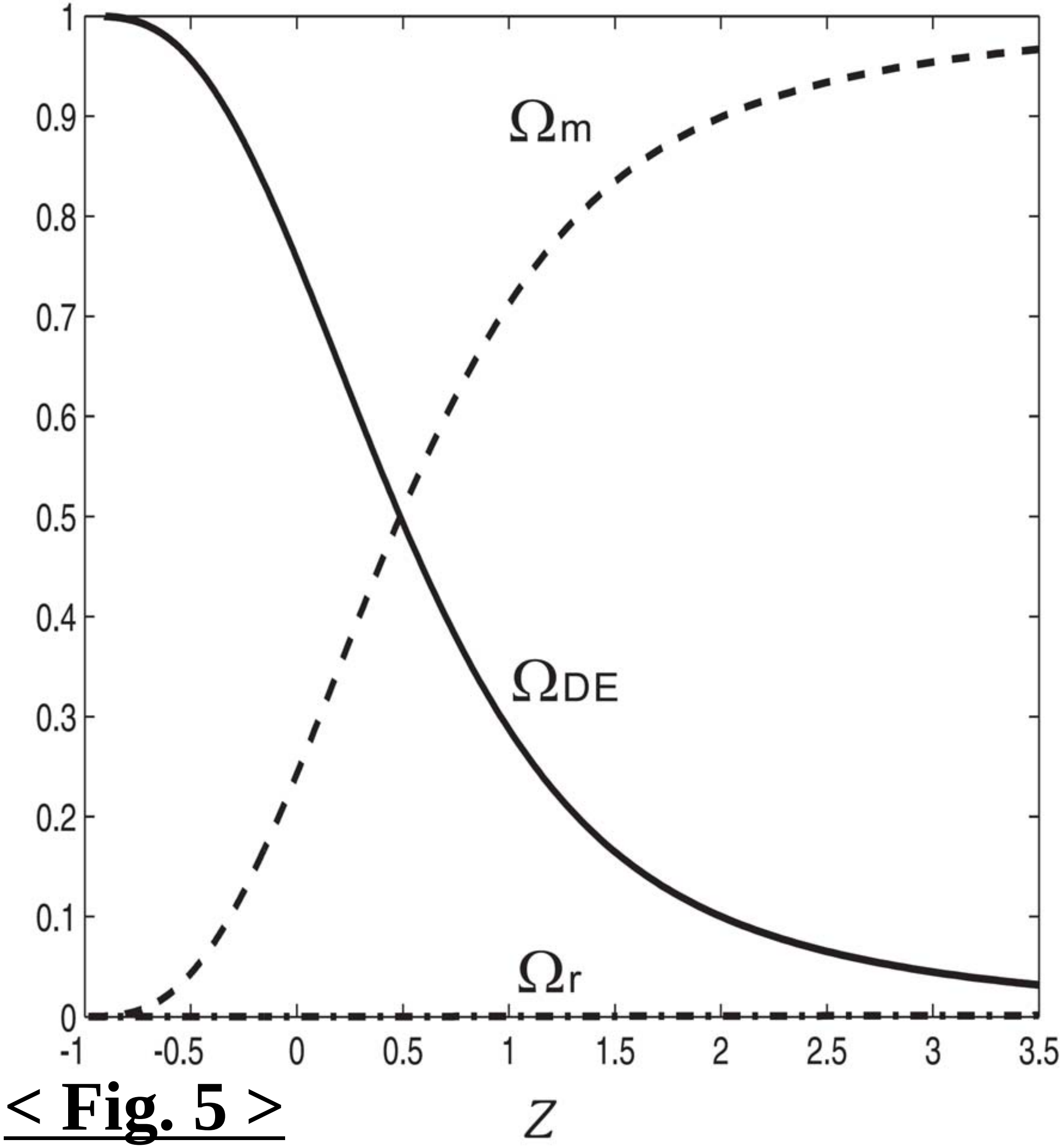
$0 < \underline{v < 10^{-10}}$: Constant parameter (close to 0)

[Capozziello and Tsujikawa, Phys. Rev. D 77, 107501 (2008)]

< Cosmological evolutions of $\Omega_{\rm DE}$, $\Omega_{\rm m}$ and $\Omega_{\rm r}$ in the exponential gravity model >

From [KB, Geng and Lee, JCAP 1008, 021 (2010)].

$$F_{\rm E}(R) = R - \beta R_{\rm E}\left(1 - {\rm e}^{-R/R_{\rm E}}\right)$$



< Fig. 5 >

$$\beta = 1.8$$

$$\beta R_{\rm E} \simeq 18 H_0^2 \Omega_{\rm m}^{(0)}$$

$$\Omega_{\rm DE} \equiv \rho_{\rm DE}/\rho_{\rm crit}^{(0)}$$

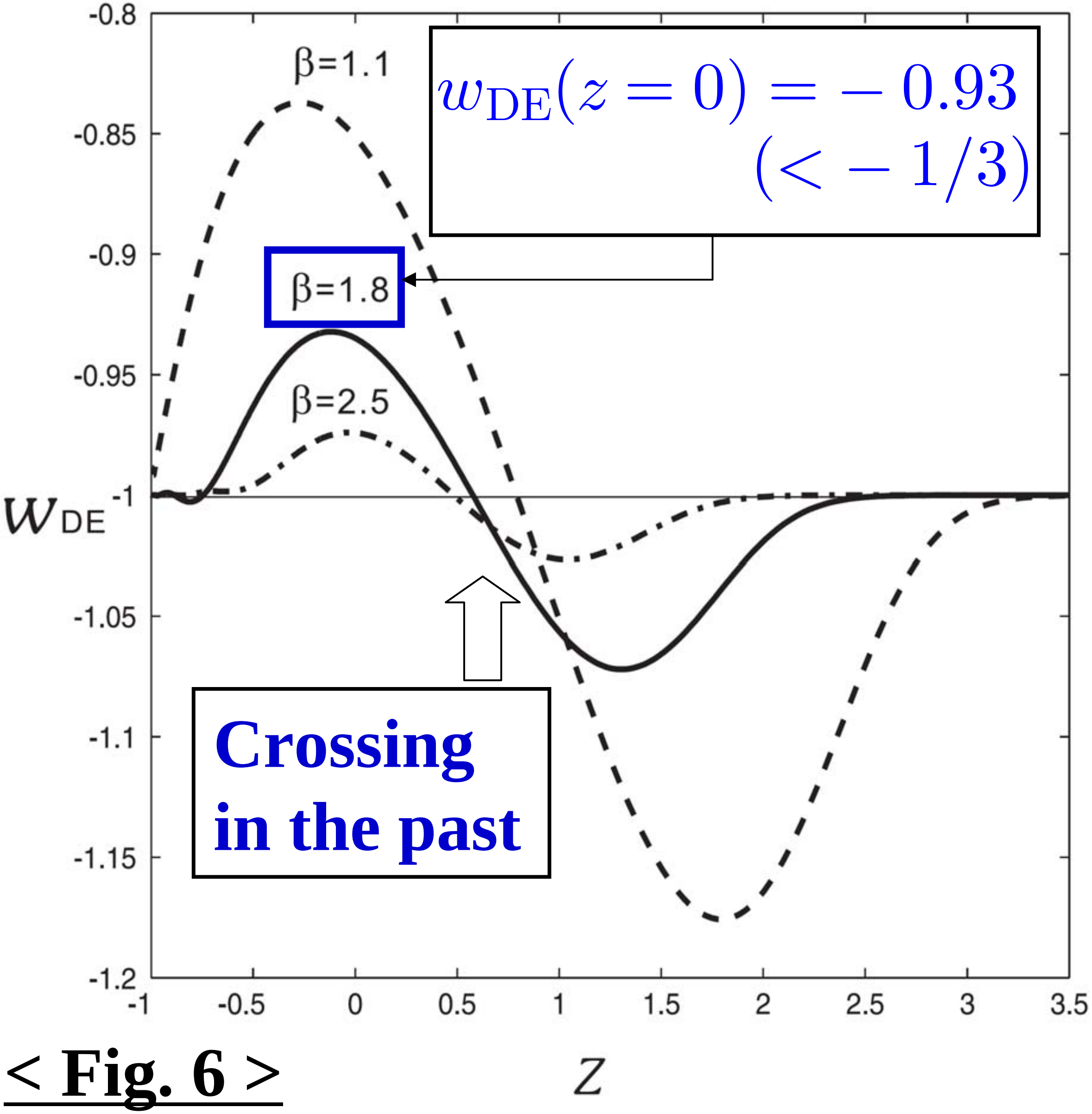
$$\Omega_{\rm m} \equiv \rho_{\rm m}/\rho_{\rm crit}^{(0)}$$

$$\Omega_{\rm r} \equiv \rho_{\rm r}/\rho_{\rm crit}^{(0)}$$

$$\rho_{\rm crit}^{(0)} = 3H_0^2/\kappa^2$$

< Cosmological evolution of $w_{\rm DE}$ in the exponential gravity model >

From [KB, Geng and Lee, JCAP 1008, 021 (2010)].



$$w_{\rm DE}(z=0) = -0.93$$
$$(< -1/3)$$

Crossing in the past

$$F_{\rm E}(R) = R - \beta R_{\rm E}\left(1 - {\rm e}^{-R/R_{\rm E}}\right)$$

$$w_{\rm DE} = -1$$

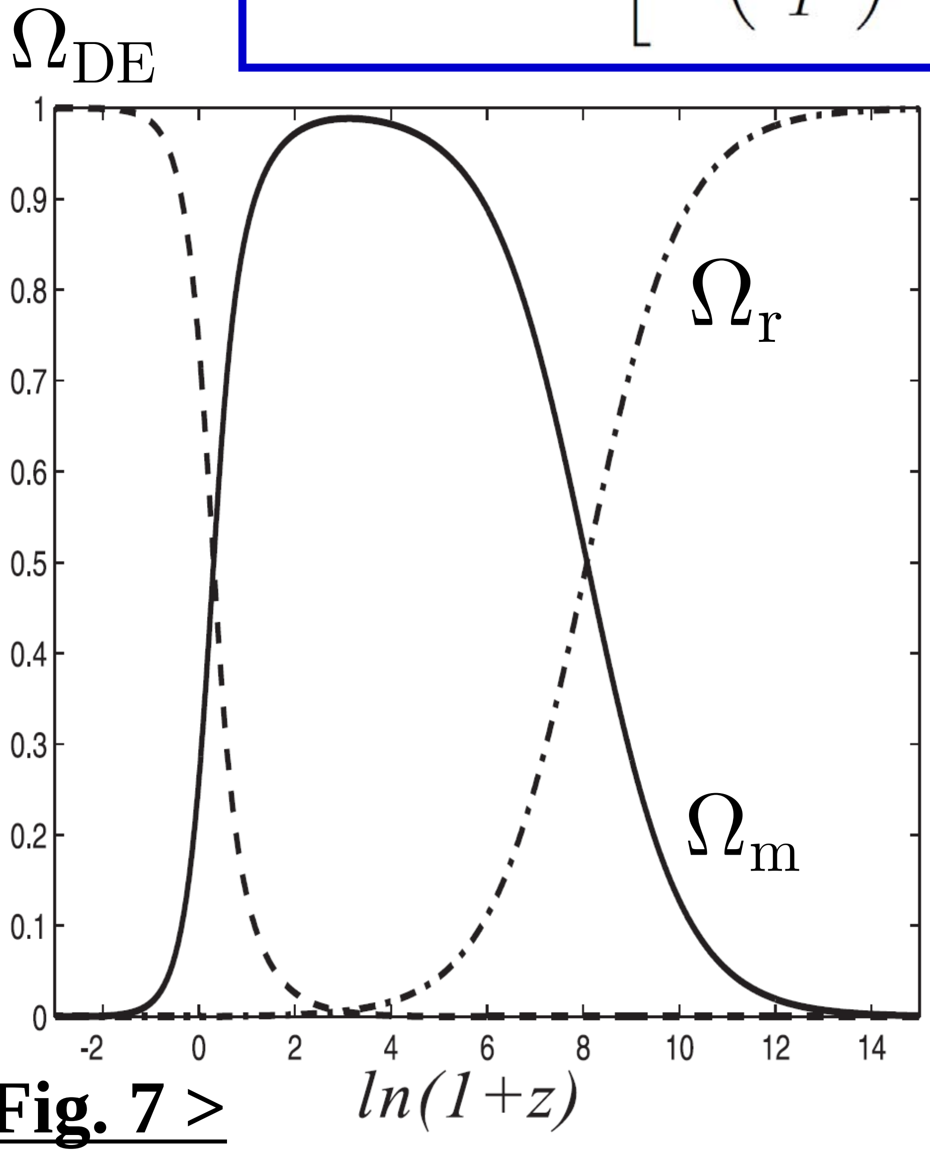
Crossing of the phantom divide

$$\beta R_{\rm E} \simeq 18 H_0^2 \Omega_{\rm m}^{(0)}$$

< Fig. 6 >

< Cosmological evolutions of Ω_{DE} , Ω_{m} and Ω_{r} >

$$\boxed{f(T) = T + \gamma \left[T_0 \left(\frac{uT_0}{T} \right)^{-1/2} \ln \left(\frac{uT_0}{T} \right) - T \left(1 - e^{uT_0/T} \right) \right]}$$



< Fig. 7 >

$$\gamma \equiv \frac{1 - \Omega_{\mathrm{m}}^{(0)}}{2u^{-1/2} + [1 - (1 - 2u)\, e^{u}]}$$

$u(> 0)$: Positive constant

$\Omega_{\mathrm{m}}^{(0)} \equiv \rho_{\mathrm{m}}^{(0)} / \rho_{\mathrm{crit}}^{(0)}$, $T_0 = T(z = 0)$

$\rho_{\mathrm{crit}}^{(0)} = 3H_0^2 / \kappa^2$

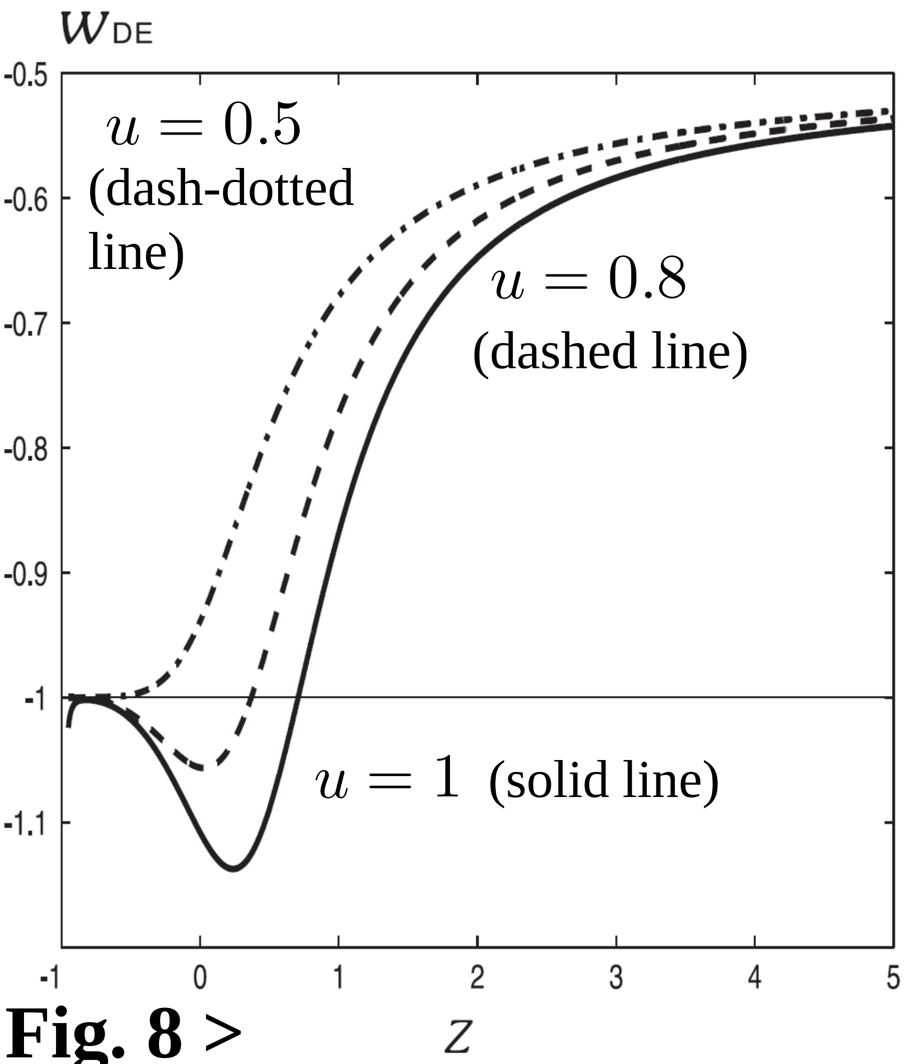
- The model contains only one parameter u if one has the value of $\Omega_{\mathrm{m}}^{(0)}$.

From [KB, Geng, Lee and Luo, JCAP 1101, 021 (2011)].

$$\boxed{u = 1}$$

< Cosmological evolutions of $w_{\rm DE}$ >

$$\boxed{f(T) = T + \gamma \left[T_0 \left(\frac{uT_0}{T} \right)^{-1/2} \ln \left(\frac{uT_0}{T} \right) - T \left(1 - e^{uT_0/T} \right) \right]}$$



< Fig. 8 >

← $w_{\rm DE} = -1$

$$\gamma \equiv \frac{1 - \Omega_{\rm m}^{(0)}}{2u^{-1/2} + \left[1 - (1 - 2u)\, e^{u} \right]}$$

$u (> 0)$: Positive constant

$\Omega_{\rm m}^{(0)} \equiv \rho_{\rm m}^{(0)} / \rho_{\rm crit}^{(0)}$, $T_0 = T(z = 0)$

$\rho_{\rm crit}^{(0)} = 3H_0^2 / \kappa^2$

From [KB, Geng, Lee and Luo, JCAP 1101, 021 (2011)].

Crossing of the phantom divide

- $$w_{\mathrm{eff}} \approx w_{\mathrm{DE}} = \frac{P_{\mathrm{DE}}}{\rho_{\mathrm{DE}}} = \frac{-\left[4\left(1-F-2TF'\right)\dot{H} + \left(-T-f+2TF\right)\right]}{-T-f+2TF}$$

$$\longrightarrow \quad P_{\mathrm{DE}} = -\rho_{\mathrm{DE}} + I(H,\dot{H})\,, \quad I \equiv -\frac{1}{\kappa^2}\left[2\left(1-F-2TF'\right)\dot{H}\right]$$

$$\dot{H} + \frac{\kappa^2}{2} I(H,\dot{H}) = 0 \Longrightarrow \dot{H}\left(F+2TF'\right) = 0 \Longrightarrow F + 2TF' = 0$$

$$\dot{H} \neq 0$$

Gravitational field equations

$$-f + 2TF = 0$$: Consistency condition (Friedmann equation)

$$-F - 2TF' = 0$$

Power-law model

$$\boxed{f(T) = AT^{\alpha}} \quad A(\neq 0)\,, \quad \alpha(\neq 0)$$

$$F + 2TF' = A\left(-6\right)^{\alpha-1}\left(2\alpha-1\right)H^{2(\alpha-1)} = 0$$

$$-f + 2TF = A\left(-6\right)^{\alpha}\left(2\alpha-1\right)H^{2\alpha} = 0$$

- By using $\ddot{\eta} + 3H\dot{\eta} = a^{-3}d\left(a^3\dot{\eta}\right)/dt$ and $0 = \ddot{\eta} + 3H\dot{\eta} + 6\dot{H} + 12H^2$,

$$\boxed{\eta = -\int^t \frac{1}{a^3}\left(\int^{\bar{t}} Ra^3 d\bar{t}\right) dt}$$

η_{c} : Integration constant

- We take a form of $f(\eta)$ as $\boxed{f(\eta) = f_{\mathrm{s}}\eta^{\sigma}}$.

$f_{\mathrm{s}}(\neq 0)$, $\sigma(\neq 0)$

: Non-zero constants

- By using $\ddot{\xi} + 3H\dot{\xi} = a^{-3}d\left(a^3\dot{\xi}\right)/dt$ and $0 = \ddot{\xi} + 3H\dot{\xi} - \left(6\dot{H} + 12H^2\right)f'(\eta)$,

$$\boxed{\xi = \int^t \frac{1}{a^3}\left(\int^{\bar{t}} \frac{df(\eta)}{d\eta} Ra^3 d\bar{t}\right) dt}$$

ξ_{c} : Integration constant

⇨ There are three cases.

(i) $[q > 1,\ \sigma > 0]$: $\eta \propto (t_{\mathrm{s}} - t)^{-(q-1)}$, $\xi \propto (t_{\mathrm{s}} - t)^{-(q-1)\sigma}$

(ii) $[q > 1,\ \sigma < 0]$: $\eta \propto (t_{\mathrm{s}} - t)^{-(q-1)}$, $\xi \sim \xi_{\mathrm{c}}$

(iii) $[-1 < q < 0\,,\ 0 < q < 1]$: $\eta \sim \eta_{\mathrm{c}}$, $\xi \sim \xi_{\mathrm{c}}$

→ We examine the behavior of each term of the gravitational field equations in the limit $t \to t_{\rm s}$, in particular that of the leading terms, and study the condition that an asymptotic solution can be obtained.

- For case (ii) $[q > 1,\ \sigma < 0]$, $\boxed{\xi_{\rm c} = 1}$
- For case (iii) $[-1 < q < 0\, ,\, 0 < q < 1]$,

$$\boxed{f_{\rm s} \eta_{\rm c}^{\sigma-1} \left(6\sigma - \eta_{\rm c} \right) + \xi_{\rm c} - 1 = 0}$$

⇒ the leading term vanishes in both gravitational field equations.

$$\boxed{H \sim \frac{h_{\rm s}}{(t_{\rm s} - t)^q}}$$

⇒ Thus, the expression of the Hubble parameter can be a leading-order solution in terms of $(t_{\rm s} - t)$ for the gravitational field equations in the flat FLRW space-time.

⇓

This implies that there can exist the finite-time future singularities in non-local gravity.

C. Relations between the model parameters and the property of the finite-time future singularities

- $\boxed{f(\eta) = f_{\mathrm{s}} \eta^{\sigma}}$ ⟶ f_{s} and σ characterize the theory of non-local gravity.
- $\boxed{H \sim \frac{h_{\mathrm{s}}}{(t_{\mathrm{s}} - t)^{q}}}$ ⟶ h_{s} , t_{s} and q specify the property of the finite-time future singularity.
- η_{c} and ξ_{c} determine a leading-order solution in terms of $(t_{\mathrm{s}} - t)$ for the gravitational field equations in the flat FLRW space-time.
- When $t \longrightarrow t_{\mathrm{s}}$,

∗ ρ_{eff} and P_{eff} correspond to the total energy density and pressure of the universe, respectively.

for $q > 1$, $a \longrightarrow \infty$

for $-1 < q < 0$ and $0 < q < 1$, $a \longrightarrow a_{\mathrm{s}}$

for $q > 0$, $H \longrightarrow \infty$, $\rho_{\mathrm{eff}} = 3H^2/\kappa^2 \longrightarrow \infty$

for $-1 < q < 0$, H asymptotically becomes finite and also ρ_{eff} asymptotically approaches a finite constant value ρ_{s} .

for $q > -1$, $\dot{H} \sim q h_{\mathrm{s}} (t_{\mathrm{s}} - t)^{-(q+1)} \to \infty$, $P_{\mathrm{eff}} = -\left(2\dot{H} + 3H^2\right)/\kappa^2 \to \infty$

→ Gravitational field equations in the flat FLRW background:

$$H^2 = \frac{8\pi G}{3}\left(\rho_{\mathrm{M}} + \rho_{\mathrm{DE}}\right)$$

$$\left(H^2\right)' = -8\pi G\left(\rho_{\mathrm{M}} + P_{\mathrm{M}} + \rho_{\mathrm{DE}} + P_{\mathrm{DE}}\right)$$

$$\rho_{\mathrm{DE}} = \frac{1}{16\pi G}\left(-f + 2Tf_T\right)$$

$$P_{\mathrm{DE}} = \frac{1}{16\pi G}\frac{f - Tf_T + 2T^2 f_{TT}}{1 + f_T + 2Tf_{TT}}$$

$$f_T \equiv df(T)/dT$$

$$f_{TT} \equiv d^2 f(T)/dT^2$$

* A prime denotes a derivative with respect to $\ln a$.

⇨ $$\boxed{w_{\mathrm{DE}} \equiv \frac{P_{\mathrm{DE}}}{\rho_{\mathrm{DE}}} = -1 + \frac{T'}{3T}\frac{f_T + 2Tf_{TT}}{f/T - 2f_T} = -\frac{f/T - f_T + 2Tf_{TT}}{\left(1 + f_T + 2Tf_{TT}\right)\left(f/T - 2f_T\right)}}$$

* We consider only non-relativistic matter (cold dark matter and baryon) with $\rho_{\mathrm{M}} = \rho_{\mathrm{m}}$ and $P_{\mathrm{M}} = P_{\mathrm{m}} = 0$.

→ Continuity equation: $$\frac{d\rho_{\mathrm{DE}}}{dN} \equiv \rho'_{\mathrm{DE}} = -3\left(1 + w_{\mathrm{DE}}\right)\rho_{\mathrm{DE}} \qquad N \equiv \ln a$$

< Combined *f*(*T*) model >

From [KB, Geng, Lee and Luo, JCAP 1101, 021 (2011)].

$$f(T) = \gamma \left[\underline{T_0 \left(\frac{uT_0}{T} \right)^{-1/2} \ln \left(\frac{uT_0}{T} \right)} \underline{- T \left(1 - e^{uT_0/T} \right)} \right]$$

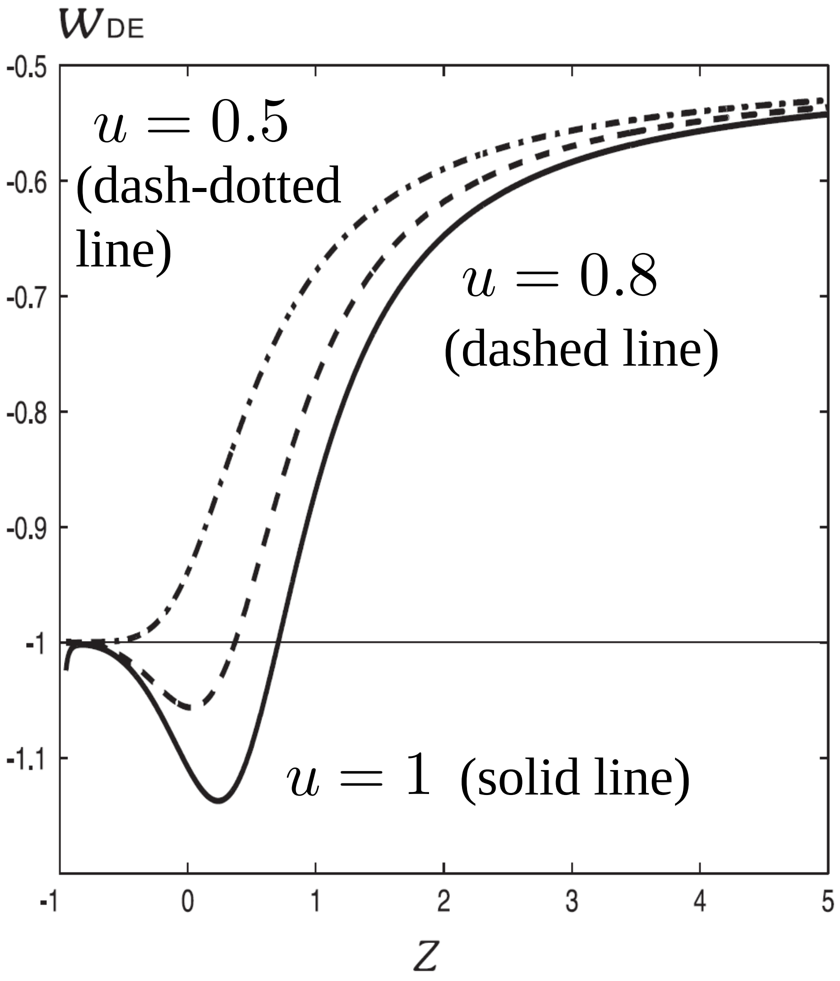
Logarithmic term

Exponential term

$u (> 0)$

: Positive constant

$T_0 = T(z = 0)$



← $w_{\mathrm{DE}} = -1$

$$\gamma \equiv \frac{1 - \Omega_{\mathrm{m}}^{(0)}}{2u^{-1/2} + \left[1 - (1 - 2u)\, e^{u} \right]}$$

$$\Omega_{\mathrm{m}}^{(0)} \equiv \rho_{\mathrm{m}}^{(0)} / \rho_{\mathrm{crit}}^{(0)}$$

$$\rho_{\mathrm{crit}}^{(0)} = 3H_0^2 / (8\pi G)$$

Crossing of the phantom divide

- The model contains only one parameter u if one has the value of $\Omega_{\mathrm{m}}^{(0)}$.

Cosmological consequences of adding an R^2 term

→ We explore whether the addition of $\underline{\text{an } R^2 \text{ term}}$ removes the finite-time future singularities in non-local gravity.

< Action >

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}\left[R\left(1+f(\Box^{-1}R)\right)\underline{+uR^2}-2\Lambda\right]+\mathcal{L}_{\text{matter}}\left(Q;g\right)\right\}$$

- Gravitational field equations in the flat FLRW background: $u(\neq 0)$

$$0 = -3H^2\left(1+f(\eta)-\xi\right)+\frac{1}{2}\dot{\xi}\dot{\eta}-3H\left(f'(\eta)\dot{\eta}-\dot{\xi}\right)\underline{+\Theta}+\Lambda+\kappa^2\rho_{\text{m}}$$

$$0 = \left(2\dot{H}+3H^2\right)\left(1+f(\eta)-\xi\right)+\frac{1}{2}\dot{\xi}\dot{\eta}+\left(\frac{d^2}{dt^2}+2H\frac{d}{dt}\right)\left(f(\eta)-\xi\right)\underline{+\Xi}-\Lambda+\kappa^2P_{\text{m}}$$

- In the limit $t \longrightarrow t_{\text{s}}$,

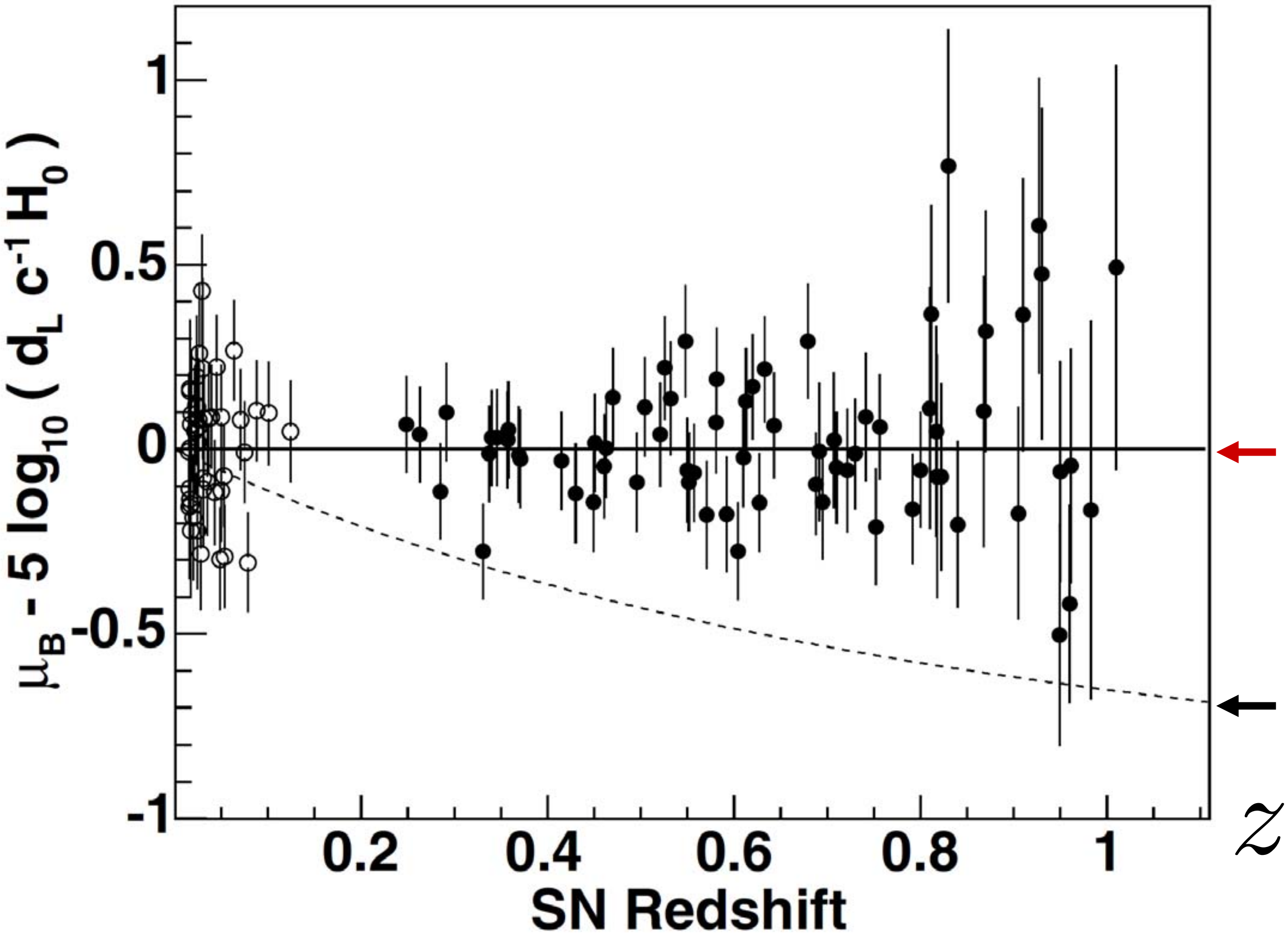
→ $$\underline{\Theta} \sim 18u\left[-6h_{\text{s}}^2q\left(t_{\text{s}}-t\right)^{-(3q+1)}+h_{\text{s}}^2q^2\left(t_{\text{s}}-t\right)^{-2(q+1)}-2h_{\text{s}}^2q\left(q+1\right)\left(t_{\text{s}}-t\right)^{-2(q+1)}\right]$$

→ $$\underline{\Xi} \sim 6u\left[9h_{\text{s}}^2q^2\left(t_{\text{s}}-t\right)^{-2(q+1)}+18h_{\text{s}}^3q\left(t_{\text{s}}-t\right)^{-(3q+1)}\right.$$
$$\left.+2h_{\text{s}}q\left(q+1\right)\left(q+2\right)\left(t_{\text{s}}-t\right)^{-(q+3)}+12h_{\text{s}}^2q\left(q+1\right)\left(t_{\text{s}}-t\right)^{-2(q+1)}\right]$$

The leading terms do not vanish. ⇨ **The additional R^2 term can remove the finite-time future singularity.**

< Residuals for the best fit to a flat Λ cosmology >

$\Delta(m - M)$



Flat Λ cosmology

Pure matter cosmology

From [Astier *et al*. [The SNLS Collaboration], Astron. Astrophys. 447, 31 (2006)]

IV. Effective equation of state for the universe and phantom-divide crossing

A. Cosmological evolution of the effective equation of state for the universe

- The effective equation of state for the universe :

$$\boxed{w_{\mathrm{eff}} \equiv \frac{P_{\mathrm{eff}}}{\rho_{\mathrm{eff}}} = -1 - \frac{2\dot{H}}{3H^2}}$$

$$\rho_{\mathrm{eff}} = \frac{3H^2}{\kappa^2}\,, \quad P_{\mathrm{eff}} = -\frac{2\dot{H} + 3H^2}{\kappa^2}$$

$\dot{H} < 0$: **The non-phantom (quintessence) phase**

$$\longrightarrow \quad w_{\mathrm{eff}} > -1$$

$\dot{H} = 0 \longrightarrow \quad w_{\mathrm{eff}} = -1$ **Phantom crossing**

$\dot{H} > 0$: **The phantom phase**

$$\longrightarrow \quad w_{\mathrm{eff}} < -1$$

→ We examine the asymptotic behavior of w_{eff} in the limit $t \to t_{\mathrm{s}}$ by taking the leading term in terms of $(t_{\mathrm{s}} - t)$.

- For $q > 1$ [Type I ("Big Rip") singularity], w_{eff} evolves from the non-phantom phase or the phantom one and asymptotically approaches $w_{\mathrm{eff}} = -1$.

- For $0 < q < 1$ [Type III singularity], w_{eff} evolves from the non-phantom phase to the phantom one with realizing a crossing of the phantom divide or evolves in the phantom phase.

  ⇨ The final stage is the eternal phantom phase.

- For $-1 < q < 0$ [Type II ("sudden") singularity], $w_{\mathrm{eff}} > 0$ at the final stage.

→ We estimate the present value of $w_{\rm eff}$.

∗ We regard $w_{\rm eff} \approx w_{\rm DE}$ at the present time because the energy density of dark energy is dominant over that of non-relativistic matter at the present time.

- For case (ii) $[q > 1, \ \sigma < 0]$,

$\sigma = -1$

$q = 2$

$h_{\rm s} = 1 \, [\text{GeV}]^{-1}$

$t_{\rm s} = 2 t_{\rm p}$

$H_{\rm p} = 2.1 h \times 10^{-42} \text{GeV}$

: Current value of H , $h = 0.7$ **[Freedman *et al*. [HST Collaboration], Astrophys. J. <u>553</u>, 47 (2001)]**

$$\boxed{\begin{array}{l} f_{\rm s} = -3.0 \times 10^{-43} \\ \underline{w_{\rm eff} = -1.10} \end{array}} \qquad \boxed{\begin{array}{l} f_{\rm s} = -2.1 \times 10^{-43} \\ \underline{w_{\rm eff} = -0.93} \end{array}}$$

$t_{\rm p}$: The present time

$h_{\rm s}$ has the dimension of $[\text{Mass}]^{q-1}$.

- For $0 < q < 1$,

$q = 1/2$

$h_{\rm s} = 1 \, [\text{GeV}]^{1/2}$

$\eta_{\rm c} = 1$

$t_{\rm s} = 2 t_{\rm p}$

$$\boxed{\begin{array}{l} f_{\rm s} = 7.9 \times 10^{-2} \\ \underline{w_{\rm eff} = -1.10} \end{array}} \qquad \boxed{\begin{array}{l} f_{\rm s} = 6.6 \times 10^{-2} \\ \underline{w_{\rm eff} = -0.93} \end{array}}$$

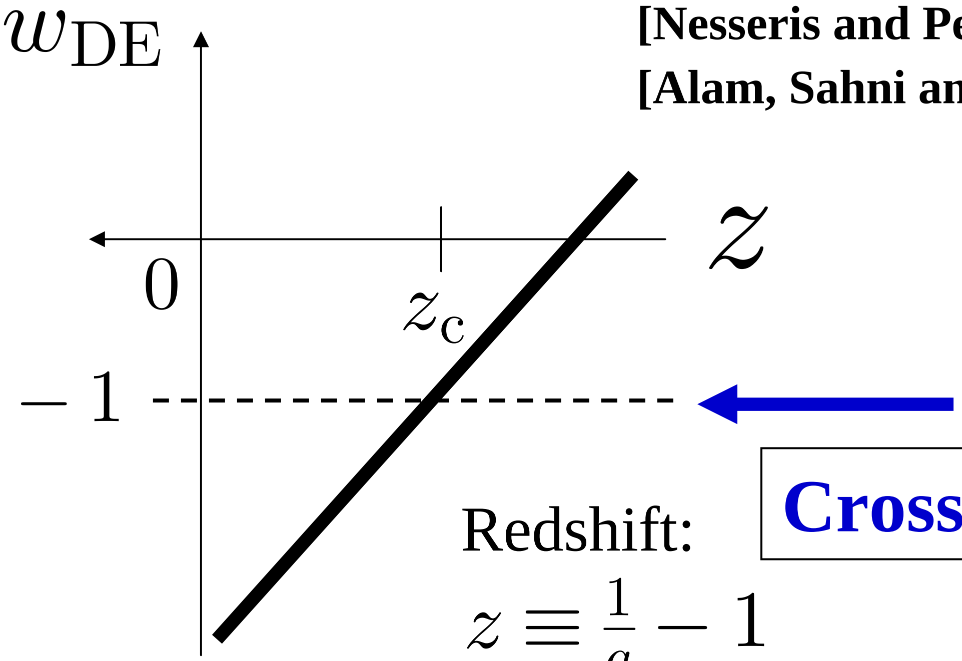
- For $-1 < q < 0$, $w_{\rm eff} > 0$.

⇨ **In our models, $w_{\rm eff}$ can have the present observed value of $w_{\rm DE}$.**

< Crossing of the phantom divide >

- Various observational data (SN, Cosmic microwave background radiation (CMB), BAO) imply that the effective EoS of dark energy $w_{\rm DE}$ may evolve from larger than -1 (non-phantom phase) to less than -1 (phantom phase). Namely, it crosses -1 (the crossing of the phantom divide).

[Alam, Sahni and Starobinsky, JCAP 0406, 008 (2004)]
[Nesseris and Perivolaropoulos, JCAP 0701, 018 (2007)]
[Alam, Sahni and Starobinsky, JCAP 0702, 011 (2007)]



Redshift: $z \equiv \frac{1}{a} - 1$

$z_{\rm c}$: Red shift at the crossing of the phantom divide

(a) $w_{\rm DE} > -1$

Non-phantom phase

(b) $w_{\rm DE} = -1$

Crossing of the phantom divide

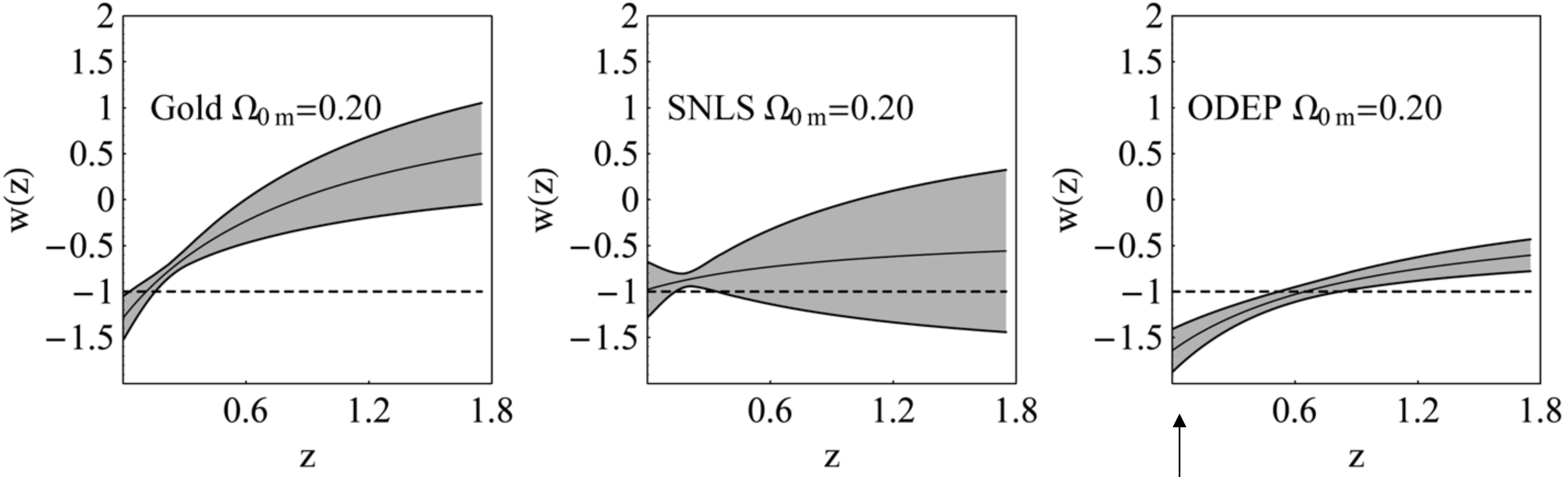
(c) $w_{\rm DE} < -1$

Phantom phase

< Data fitting of $w(z)$ >

$$w(z) = w_0 + w_1 \frac{z}{1+z}$$

From [Nesseris and L. Perivolaropoulos, JCAP 0701, 018 (2007)].



SN gold data set

[Riess *et al*. [Supernova Search Team Collaboration], Astrophys. J. 607, 665 (2004)]

SNLS data set

[Astier *et al*. [The SNLS Collaboration], Astron. Astrophys. 447, 31 (2006)]

Shaded region shows 1σ error.

Cosmic microwave background radiation (CMB) data

[Spergel *et al*. [WMAP Collaboration], Astrophys. J. Suppl. 170, 377 (2007)]

+ SDSS baryon acoustic peak (BAO) data

[Eisenstein *et al*. [SDSS Collaboration], Astrophys. J. 633, 560 (2005)]

→ Continuity equation: $\frac{d\rho_{\mathrm{DE}}}{dN} \equiv \rho'_{\mathrm{DE}} = -3\left(1 + w_{\mathrm{DE}}\right)\rho_{\mathrm{DE}}$

$N \equiv \ln a$

- We define a dimensionless variable : $\boxed{y_H \equiv \frac{H^2}{\bar{m}^2} - a^{-3} = \frac{\rho_{\mathrm{DE}}}{\rho_{\mathrm{m}}^{(0)}}}$

$\bar{m}^2 \equiv \frac{8\pi G \rho_{\mathrm{m}}^{(0)}}{3}$

$\boxed{y'_H = -3\left(1 + w_{\mathrm{DE}}\right) y_H}$: Evolution equation of the universe

< (a). Exponential *f*(*T*) theory >

$$\boxed{f(T) = \alpha T\left(1 - e^{pT_0/T}\right)}$$

$$\alpha = -\frac{1 - \Omega_{\mathrm{m}}^{(0)}}{1 - (1 - 2p)\, e^{p}}$$

p : Constant

$T_0 = T(z = 0)$

$\Omega_{\mathrm{m}}^{(0)} \equiv \rho_{\mathrm{m}}^{(0)} / \rho_{\mathrm{crit}}^{(0)}$

$\rho_{\mathrm{crit}}^{(0)} = 3H_0^2 / \left(8\pi G\right)$

- The case in which $p = 0$ corresponds to the Λ CDM model.
- This theory contains only one parameter p if the value of $\Omega_{\mathrm{m}}^{(0)}$ is given.

< (c). Combined *f(T)* theory >

$$f(T) = \gamma \left[\underline{T_0 \left(\frac{uT_0}{T} \right)^{-1/2} \ln \left(\frac{uT_0}{T} \right)} \underline{- T \left(1 - e^{uT_0/T} \right)} \right]$$

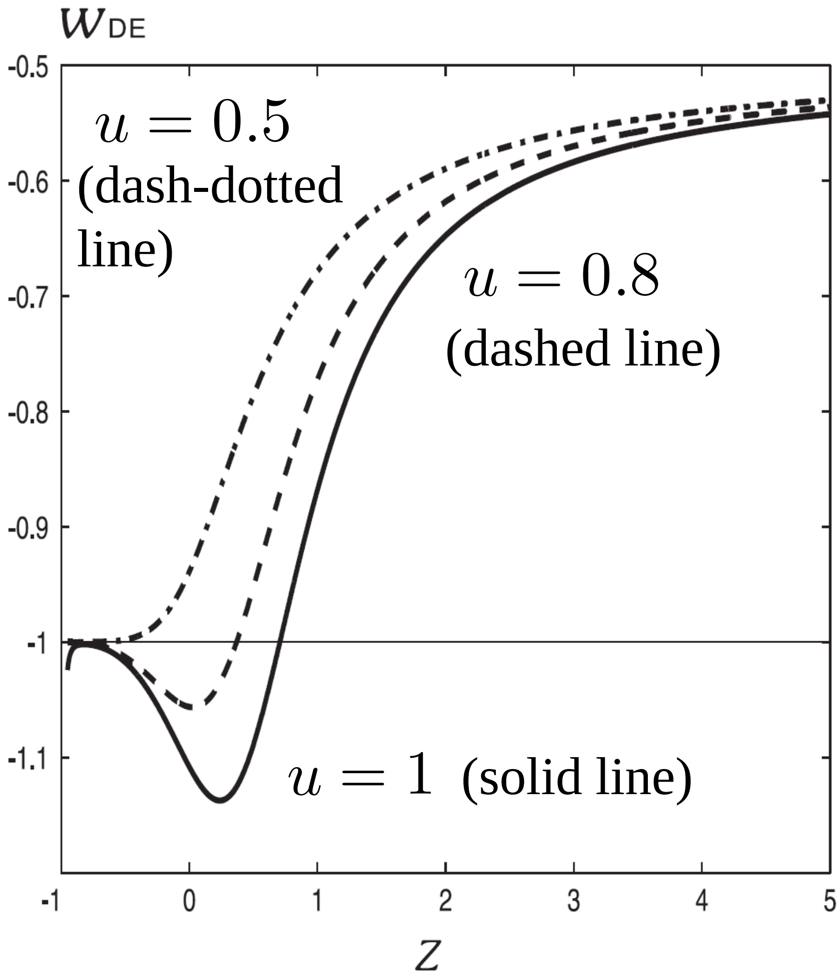
$u (> 0)$

: Positive constant

($p = q = u > 0$)

Logarithmic term

Exponential term



← $w_{\mathrm{DE}} = -1$

Crossing of the phantom divide

$$\gamma \equiv \frac{1 - \Omega_{\mathrm{m}}^{(0)}}{2u^{-1/2} + \left[1 - \left(1 - 2u \right) e^{u} \right]}$$

- The model contains only one parameter u if one has the value of $\Omega_{\mathrm{m}}^{(0)}$.

< Conditions for the viability of *f*(*R*) gravity >

(1) $\boxed{f'(R) > 0}$

- **Positivity of the effective gravitational coupling**

$G_{\text{eff}} = G_0 / f'(R) > 0$ G_0 : Gravitational constant

(The graviton is not a ghost.)

(2) $\boxed{f''(R) > 0}$ **[Dolgov and Kawasaki, Phys. Lett. B 573, 1 (2003)]**

- **Stability condition:** $M^2 \approx 1/(3f''(R)) > 0$

M : **Mass of a new scalar degree of freedom (called the "scalaron") in the weak-field regime.**

(The scalaron is not a tachyon.)

(3) $\boxed{f(R) \to R - 2\Lambda \quad \text{for} \quad R \gg R_0}$

R_0 : Current curvature

Λ : Cosmological constant

- **Realization of the ΛCDM-like behavior in the large curvature regime**

↑ Standard cosmology [Λ + Cold dark matter (CDM)]

(4) Solar system constraints

$f(R)$ gravity $\Longleftrightarrow$ **Brans-Dicke theory with $\omega_{\mathrm{BD}} = 0$**

Equivalent

ω_{BD} : Brans-Dicke parameter

[Bertotti, Iess and Tortora, Nature 425, 374 (2003).] $\longrightarrow$ Observational constraint: $|\omega_{\mathrm{BD}}| > 40000$

[Chiba, Phys. Lett. B 575, 1 (2003)]

[Erickcek, Smith and Kamionkowski, Phys. Rev. D 74, 121501 (2006)]

[Chiba, Smith and Erickcek, Phys. Rev. D 75, 124014 (2007)]

- However, if the mass of the scalar degree of freedom M is large, namely, the scalar becomes short-ranged, it has no effect at Solar System scales.
- $\boxed{M = M(R)}$ $\leftarrow$ Scale-dependence: "**Chameleon mechanism**"

Cf. [Khoury and Weltman, Phys. Rev. D 69, 044026 (2004)]

⇨ The scalar degree of freedom may acquire a large effective mass at terrestrial and Solar System scales, shielding it from experiments performed there.

(5) Existence of a matter-dominated stage and that of a late-time cosmic acceleration

- **Combing local gravity constraints, it is shown that** $\boxed{m \equiv Rf''(R)/f'(R)}$ **has to be several orders of magnitude smaller than unity.**

- For general relativity, $m = 0$.

m quantifies the deviation from the Λ CDM model.

[Amendola, Gannouji, Polarski and Tsujikawa, Phys. Rev. D 75, 083504 (2007)]

[Amendola and Tsujikawa, Phys. Lett. B 660, 125 (2008)]

(6) Stability of the de Sitter space

$$\frac{(f_{\rm d}')^2 - 2f_{\rm d}f_{\rm d}''}{f_{\rm d}'f_{\rm d}''} > 0$$

$$f_{\rm d} = f(R_{\rm d})$$

$R_{\rm d}$: Constant curvature
in the de Sitter space

- **Linear stability of the inhomogeneous perturbations in the de Sitter space** **[Faraoni and Nadeau, Phys. Rev. D 75, 023501 (2007)]**

Cf. $R_{\rm d} = 2f_{\rm d}/f_{\rm d}'$ $\Longrightarrow$ $\boxed{m < 1}$

(4) Stability of the late-time de Sitter point

$$0 < m \equiv Rf''(R)/f'(R) < 1$$

[Amendola, Gannouji, Polarski and Tsujikawa, Phys. Rev. D 75, 083504 (2007)]

- For general relativity, $m = 0$.

[Amendola and Tsujikawa, Phys. Lett. B 660, 125 (2008)]

[Faraoni and Nadeau, Phys. Rev. D 75, 023501 (2007)]

$\longrightarrow$ m quantifies the deviation from the Λ CDM model.

(5) Constraints from the violation of the equivalence principle

$$M = M(R)$$

Scale-dependence $\longleftarrow$ **"Chameleon mechanism"**

Cf. [Khoury and Weltman, Phys. Rev. D 69, 044026 (2004)]

- If the mass of the scalar degree of freedom M is large, namely, the scalar becomes short-ranged, it has no effect at Solar System scales.

$\Rightarrow$ The scalar degree of freedom may acquire a large effective mass at terrestrial and Solar System scales, shielding it from experiments performed there.

(6) Solar-system constraints

[Chiba, Phys. Lett. B 575, 1 (2003)]

[Chiba, Smith and Erickcek, Phys. Rev. D 75, 124014 (2007)]

→ Modified Friedmann equations in the flat FLRW background:

$$H^2 = \frac{8\pi G}{3}\left(\rho_{\mathrm{M}} + \rho_{\mathrm{DE}}\right) \qquad \left(H^2\right)' = -8\pi G\left(\rho_{\mathrm{M}} + P_{\mathrm{M}} + \rho_{\mathrm{DE}} + P_{\mathrm{DE}}\right)$$

$$\rho_{\mathrm{DE}} = \frac{1}{16\pi G}\left(-f + 2Tf_T\right) \qquad f_T \equiv df(T)/dT \qquad f_{TT} \equiv d^2 f(T)/dT^2$$

$$P_{\mathrm{DE}} = \frac{1}{16\pi G}\frac{f - Tf_T + 2T^2 f_{TT}}{1 + f_T + 2Tf_{TT}}$$

A prime denotes a derivative with respect to $\ln a$.

⇨ $$\boxed{w_{\mathrm{DE}} \equiv \frac{P_{\mathrm{DE}}}{\rho_{\mathrm{DE}}} = -1 + \frac{T'}{3T}\frac{f_T + 2Tf_{TT}}{f/T - 2f_T} = -\frac{f/T - f_T + 2Tf_{TT}}{\left(1 + f_T + 2Tf_{TT}\right)\left(f/T - 2f_T\right)}}$$

We consider only non-relativistic matter (cold dark matter and baryon) with $\rho_{\mathrm{M}} = \rho_{\mathrm{m}}$ and $P_{\mathrm{M}} = P_{\mathrm{m}} = 0.$

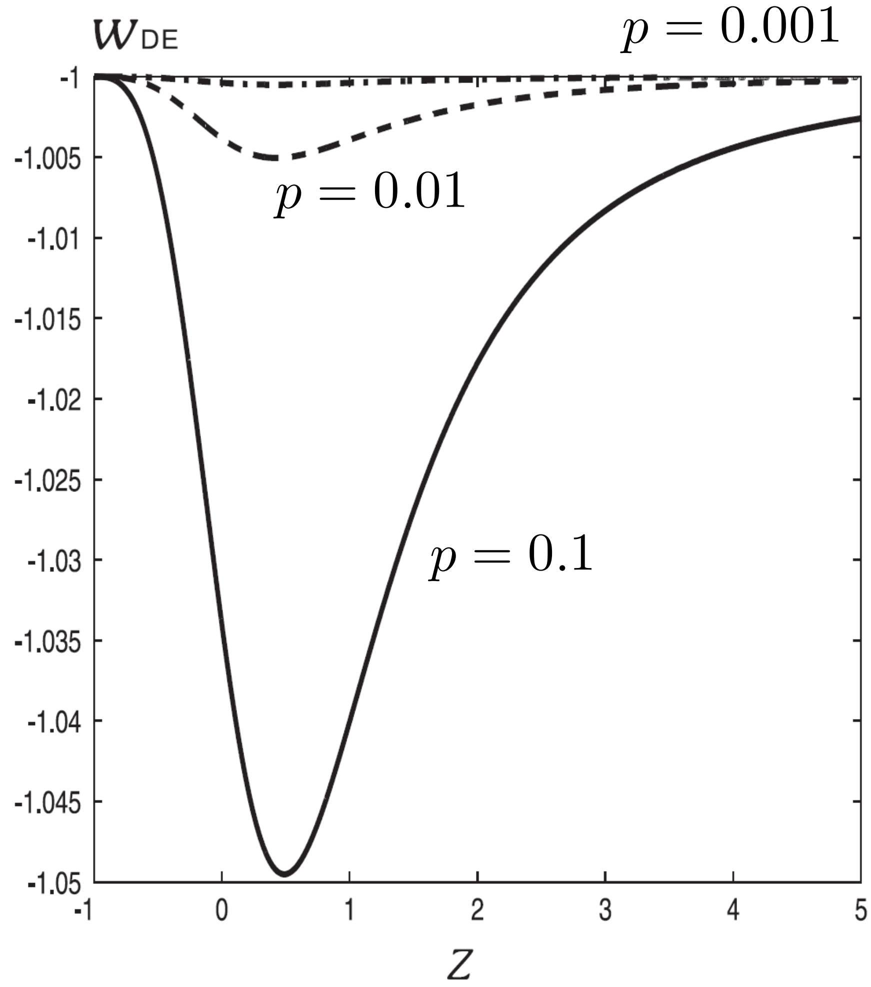
→ Continuity equation: $$\frac{d\rho_{\mathrm{DE}}}{dN} \equiv \rho'_{\mathrm{DE}} = -3\left(1 + w_{\mathrm{DE}}\right)\rho_{\mathrm{DE}}$$

- We define a dimensionless variable : $$\boxed{y_H \equiv \frac{H^2}{\bar{m}^2} - a^{-3} = \frac{\rho_{\mathrm{DE}}}{\rho_{\mathrm{m}}^{(0)}}}$$ $$\bar{m}^2 \equiv \frac{8\pi G\rho_{\mathrm{m}}^{(0)}}{3}$$

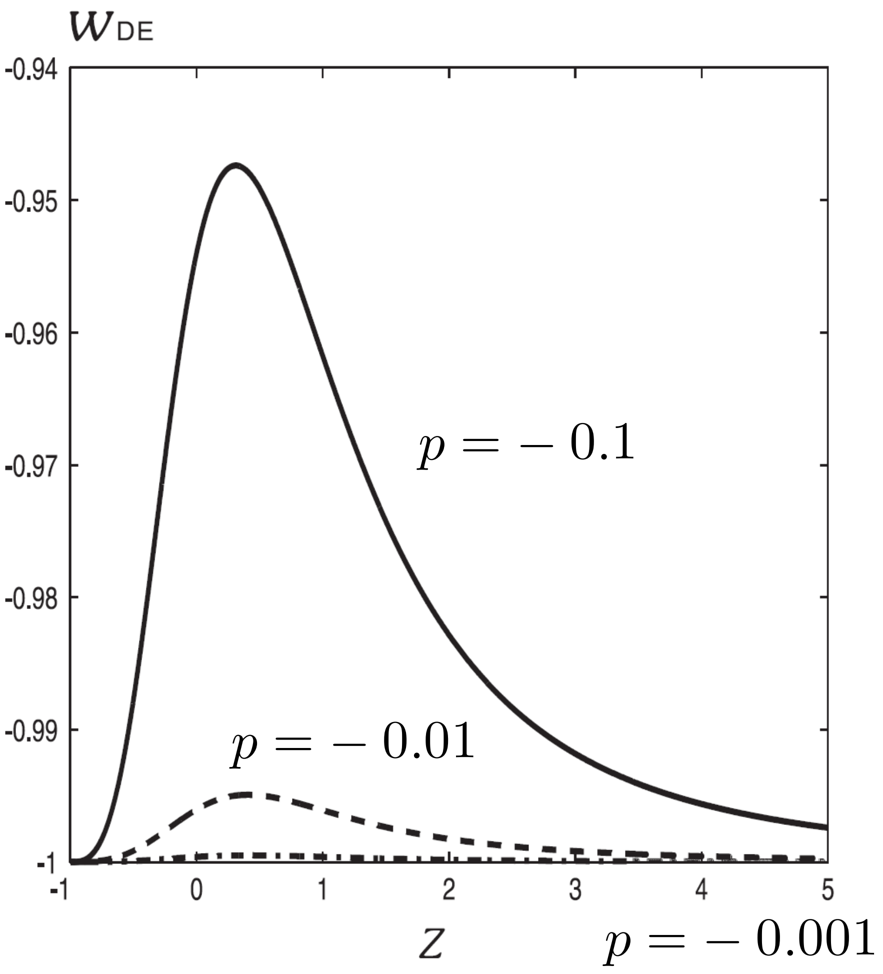
$$\boxed{y'_H = -3\left(1 + w_{\mathrm{DE}}\right)y_H}$$: Evolution equation of the universe

<(a). Exponential *f*(*T*) theory >

$$f(T) = \alpha T \left(1 - e^{pT_0/T}\right)$$



$p > 0$



$p < 0$

- $|p| = 0.1$ (solid line), 0.01 (dashed line), 0.001 (dash-dotted line)
- $\Omega_{\mathrm{m}}^{(0)} = 0.26$

< Equation of state for (the component corresponding to) dark energy >

$$\boxed{w_{\rm DE} \equiv \frac{P_{\rm DE}}{\rho_{\rm DE}}}$$

$$\rho_{\rm DE} = \frac{1}{\kappa^2}\left[\frac{1}{2}\left(FR - f\right) - 3H\dot{F} + 3\left(1 - F\right)H^2\right]$$

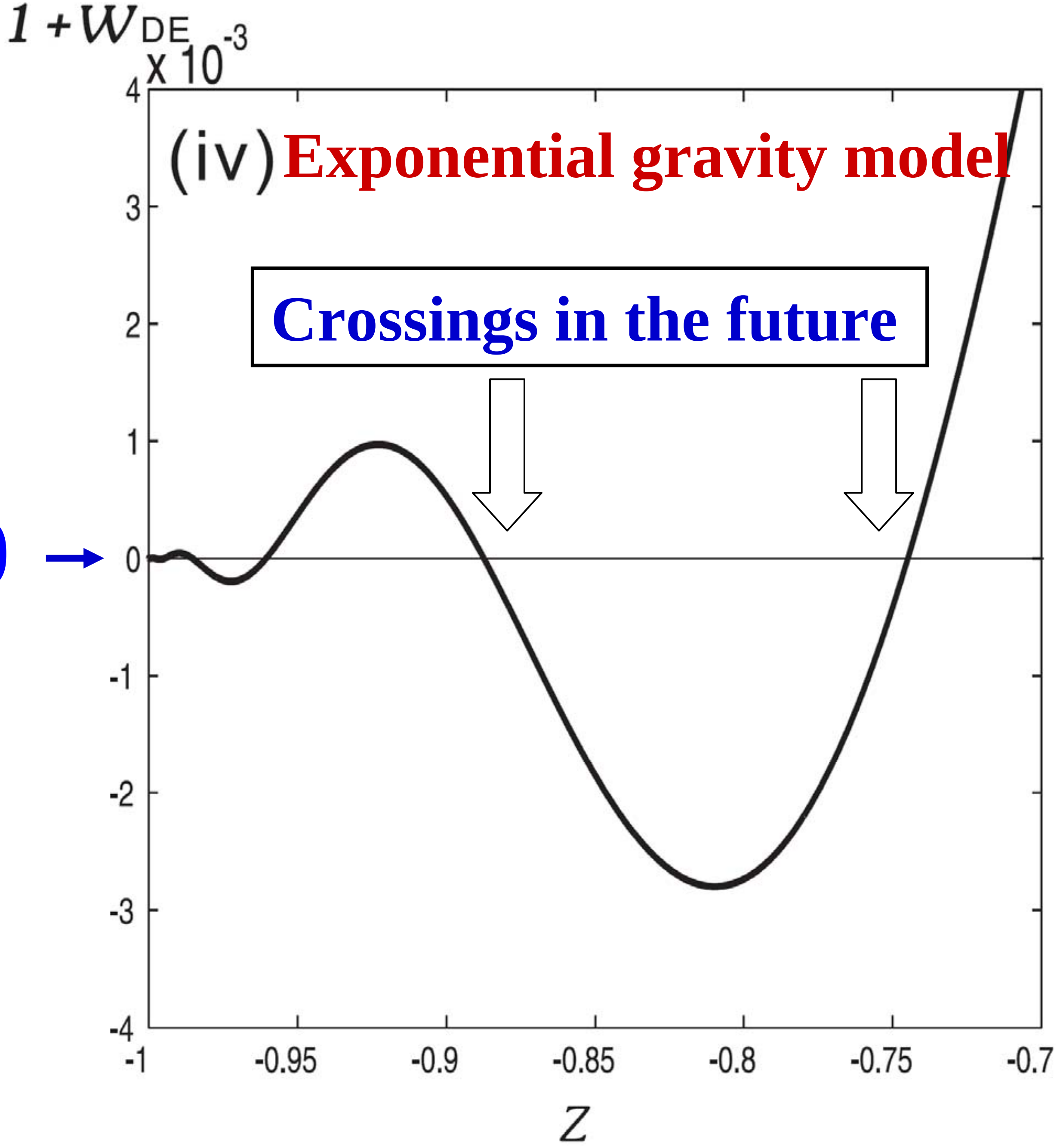
$$P_{\rm DE} = \frac{1}{\kappa^2}\left[-\frac{1}{2}\left(FR - f\right) + \ddot{F} + 2H\dot{F} - \left(1 - F\right)\left(2\dot{H} + 3H^2\right)\right]$$

< Continuity equation for dark energy >

$$\dot{\rho}_{\rm DE} + 3H\left(1 + w_{\rm DE}\right)\rho_{\rm DE} = 0$$

$$\Rightarrow \quad \boxed{w_{\rm DE} = -1 - \frac{1}{3}\frac{1}{y_H}\frac{dy_H}{d\ln a}}$$

< Future evolutions of $1 + w_{\mathrm{DE}}$ as functions of z >



Exponential gravity model

Crossings in the future

0

$$\boxed{1 + w_{\mathrm{DE}}}$$

$$1 + w_{\mathrm{DE}} = 0$$

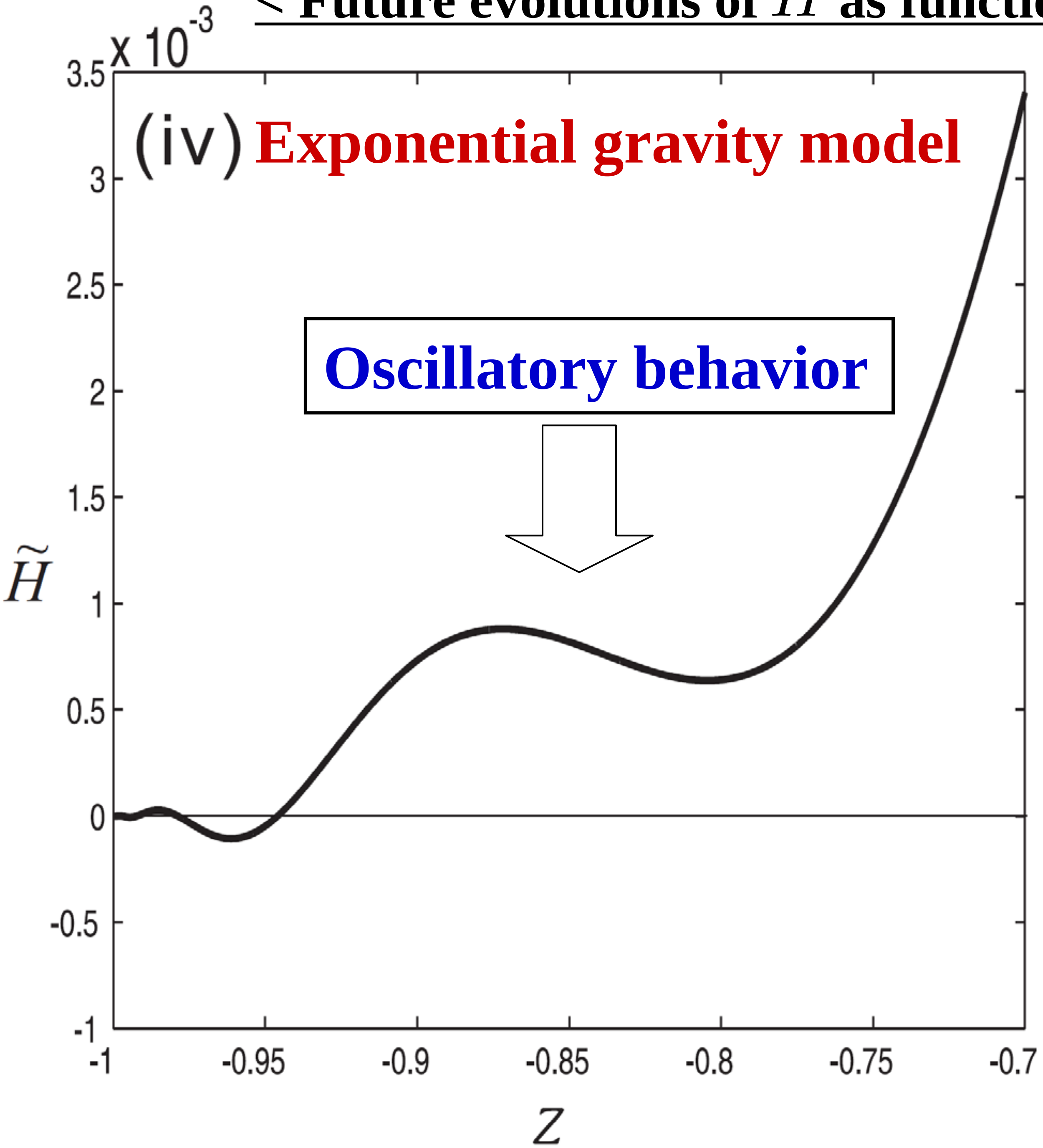
Crossing of the phantom divide

Redshift:

$$z \equiv \frac{1}{a} - 1$$

($z < 0$: Future)

< Future evolutions of H as functions of z >



$$\boxed{\tilde{H} \equiv \bar{H} - \bar{H}_{\mathrm{f}}}$$

$$\bar{H} \equiv H / H_0$$

$$\bar{H}_{\mathrm{f}} \equiv \frac{H(z=-1)}{H_0}$$

: ‘f’ denotes the value at the final stage $z = -1.$

$$H_0 = 71 \mathrm{km/s/Mpc}$$

: Present value of the Hubble parameter

- In the future ($-1 \leq z \lesssim -0.74$), the crossings of the phantom divide are the generic feature for all the existing viable *f*(*R*) models.
- As z decreases ($-1 \leq z \lesssim -0.90$), dark energy becomes much more dominant over non-relativistic matter ($\underline{\Xi = \Omega_{\mathrm{m}}/\Omega_{\mathrm{DE}} \lesssim 10^{-5}}$).

< Effective equation of state for the universe >

$$\boxed{w_{\mathrm{eff}} \equiv -1 - \frac{2}{3}\frac{\dot{H}}{H^2} = \frac{P_{\mathrm{tot}}}{\rho_{\mathrm{tot}}}}$$

$\rho_{\mathrm{tot}} \equiv \rho_{\mathrm{DE}} + \rho_{\mathrm{m}} + \rho_{\mathrm{r}}$

: Total energy density of the universe

$P_{\mathrm{tot}} \equiv P_{\mathrm{DE}} + P_{\mathrm{m}} + P_{\mathrm{r}}$

: Total pressure of the universe

$$\Rightarrow \boxed{w_{\mathrm{DE}} \approx w_{\mathrm{eff}}}$$

P_{DE} : Pressure of dark energy

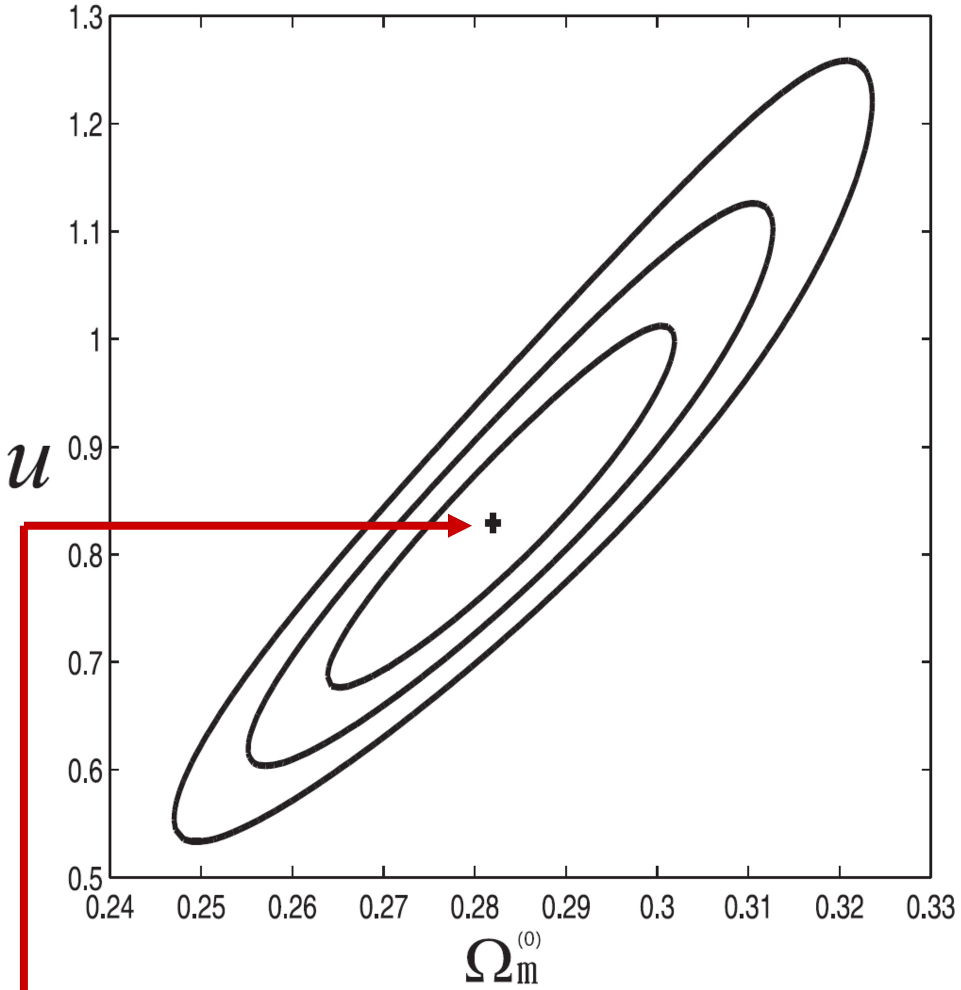
P_{m} : Pressure of non-relativistic matter (cold dark matter and baryon)

P_{r} : Pressure of radiation

< The best-fit values >

Model	u	$\Omega_{\rm m}^{(0)}$	h	$\chi^2_{\rm min}$
$f(T)$	0.829	0.282	0.691	544.56
ΛCDM		0.275	0.707	545.23

The minimum χ^2 ($\chi^2_{\rm min}$) of the combined *f*(*T*) theory is slightly smaller than that of the Λ CDM model.



The combined *f*(*T*) theory can fit the observational data well.

Contours of $68.27\%\ (1\sigma)$ **,** $95.45\%\ (2\sigma)$ **and** $99.73\%\ (3\sigma)$ **confidence levels in the** $(\Omega_{\rm m}^{(0)}, u)$ **plane from SNe Ia, BAO and CMB data.**

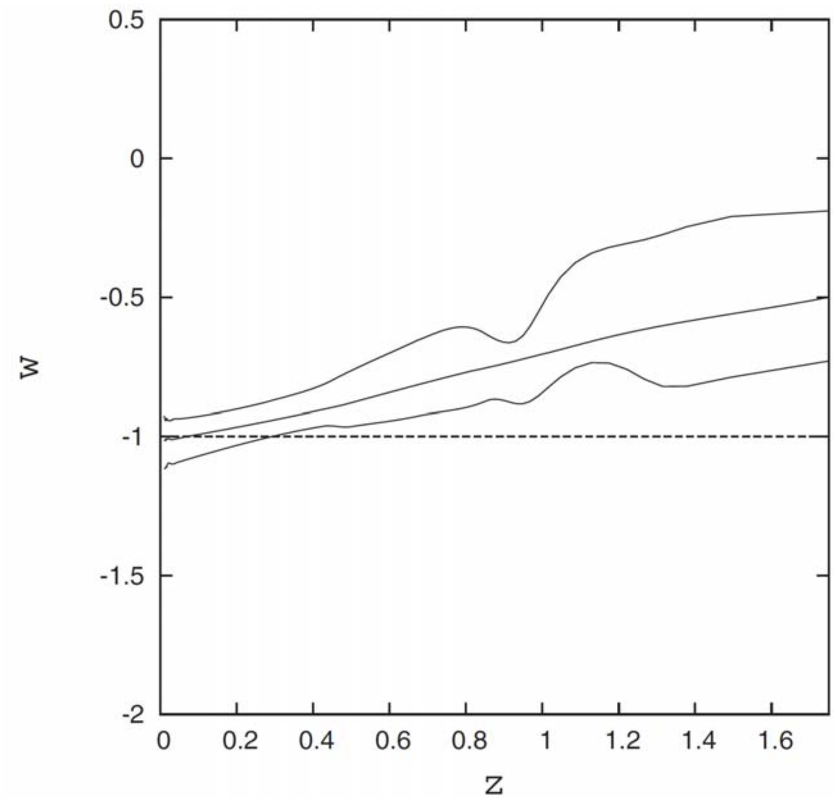
The plus sign depicts the best-fit point.

< Data fitting of $w(z)$ (2) >

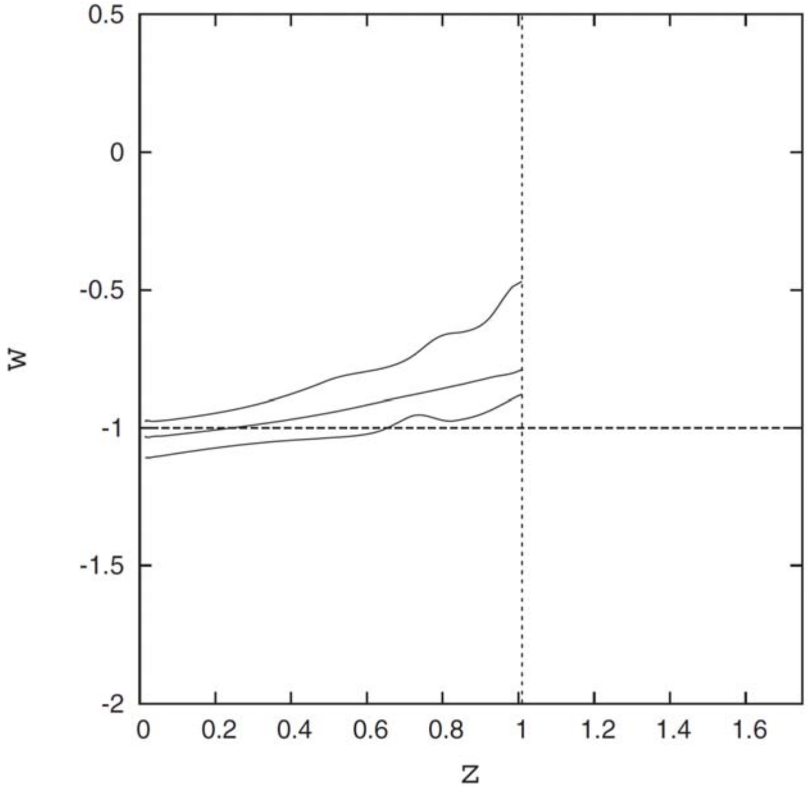
$$\boxed{w(x) = \frac{(2x/3)\,\mathrm{d}\ln H/\mathrm{d}x - 1}{1 - (H_0/H)^2 \Omega_{0m} x^3}}$$

From [Alam, Sahni and Starobinsky, JCAP 0702, 011 (2007)].

$$x = 1 + z$$



SN gold data set+CMB+BAO



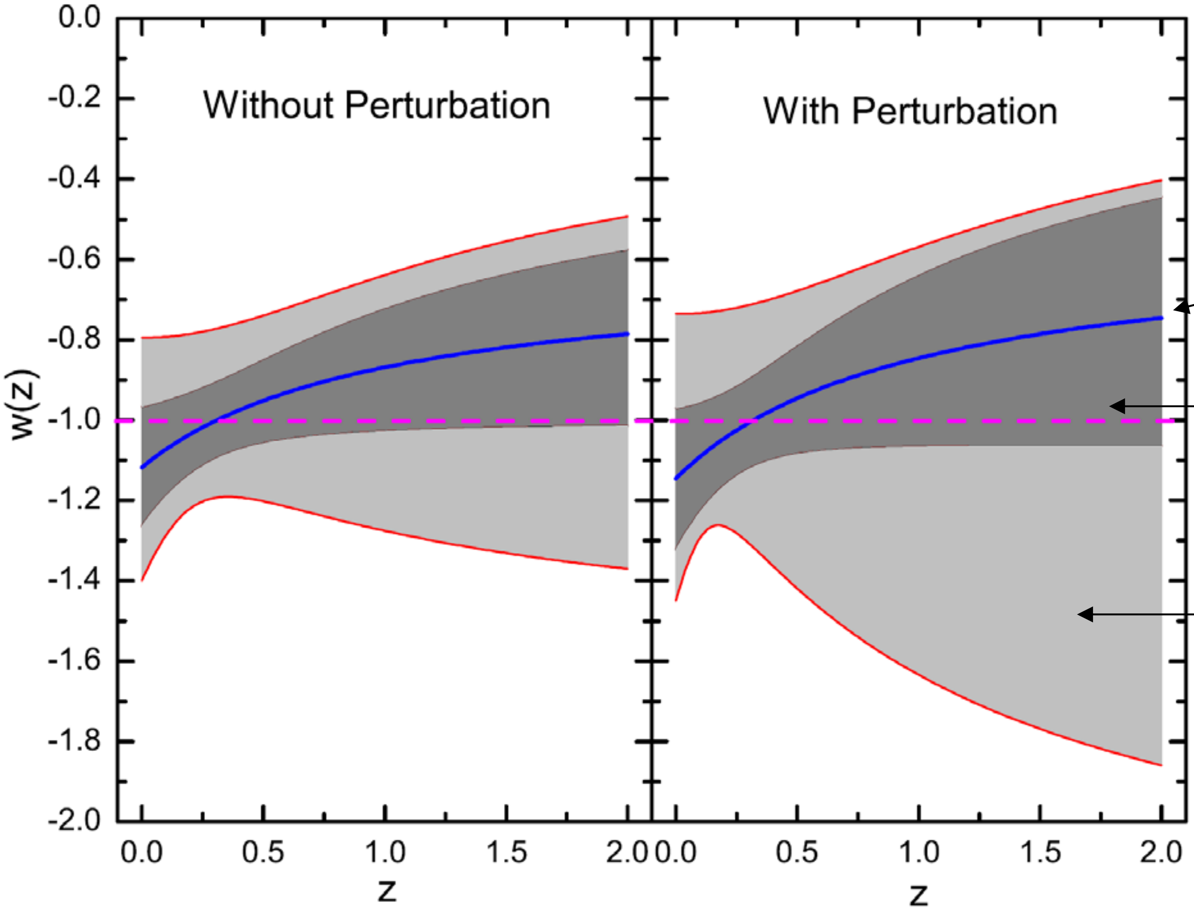
SNLS data set+CMB+BAO

- $\Omega_{0\mathrm{m}} = 0.28 \pm 0.03$
- 2σ **confidence level.**

< Data fitting of $w(z)$ (3) >

$$\boxed{w(z) = w_0 + w_1 \frac{z}{1+z}}$$

From [Zhao, Xia, Feng and Zhang, Int. J. Mod. Phys. D 16, 1229 (2007) [arXiv:astro-ph/0603621]]



Best-fit

68% **confidence level**

95% **confidence level**

157 “gold” SN Ia data set+WMAP 3-year data+SDSS with/without dark energy perturbations.

IV. Effective equation of state for the universe and the finite-time future singularities in non-local gravity

Non-local gravity ⇐ **produced by quantum effects**

[Deser and Woodard, Phys. Rev. Lett. 99, 111301 (2007)]

- There was a proposal on the solution of the cosmological constant problem by non-local modification of gravity.

[Arkani-Hamed, Dimopoulos, Dvali and Gabadadze, arXiv:hep-th/0209227]

→ Recently, an explicit mechanism to screen a cosmological constant in non-local gravity has been discussed.

[Nojiri, Odintsov, Sasaki and Zhang, Phys. Lett. B 696, 278 (2011)]

Recent related reference: [Zhang and Sasaki, arXiv:1108.2112 [gr-qc]]

- It is known that so-called matter instability occurs in $F(R)$ gravity.

[Dolgov and Kawasaki, Phys. Lett. B 573, 1 (2003)]

→ This implies that the curvature inside matter sphere becomes very large and hence the curvature singularity could appear.

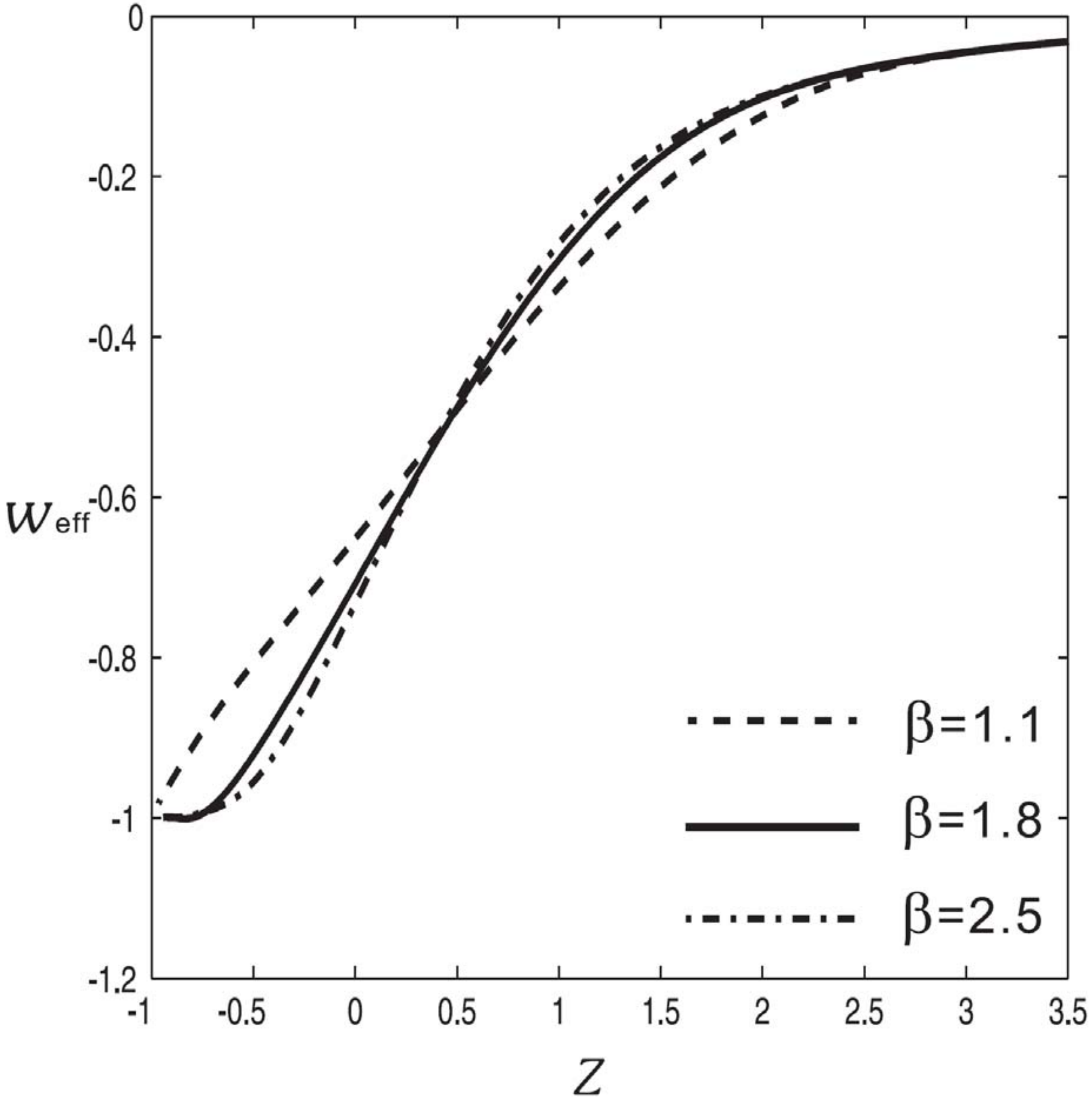
[Arbuzova and Dolgov, Phys. Lett. B 700, 289 (2011)]

⇒ It is important to examine whether there exists the curvature singularity, i.e., "**the finite-time future singularities**" **in non-local gravity**.

Appendix

< Cosmological evolution of $w_{\rm eff}$ in the exponential gravity model >

From [KB, Geng and Lee, JCAP 1008, 021 (2010)].



$$f_{\rm E}(R) = R - \beta R_{\rm E}\left(1 - e^{-R/R_{\rm E}}\right)$$

$$\beta R_{\rm E} \simeq 18 H_0^2 \Omega_{\rm m}^{(0)}$$

< 5-year WMAP data on w >

[Komatsu *et al*. [WMAP Collaboration], Astrophys. J. Suppl. 180, 330 (2009), arXiv:0803.0547 [astro-ph]]

- For the flat universe, constant w : (From WMAP+BAO+SN)

$$-0.14 < 1 + w < 0.12 \quad (95\% \text{ CL})$$

Baryon acoustic oscillation (BAO) : Special pattern in the large-scale correlation function of Sloan Digital Sky Survey (SDSS) luminous red galaxies

- For a variable EoS :

$$-0.33 < 1 + w_0 < 0.21 \quad (95\% \text{ CL}) \longleftarrow z_{\text{trans}} = 10$$

$$w(a) = \frac{a\tilde{w}(a)}{a + a_{\text{trans}}} - \frac{a_{\text{trans}}}{a + a_{\text{trans}}}$$

$a < a_{\text{trans}}$: Dark energy density tends to a constant value

$$\tilde{w}(a) = \tilde{w}_0 + (1 - a)\tilde{w}_a \quad w_0 = w(a = 1)$$

Cf. Dark Energy : $\Omega_\Lambda = 0.726 \pm 0.015$

Dark Matter : $\Omega_c = 0.228 \pm 0.013$

Baryon : $\Omega_b = 0.0456 \pm 0.0015$ (68% CL)

$$\Omega_i \equiv \frac{\kappa^2 \rho_i^{(0)}}{3H_0^2} = \frac{\rho_i^{(0)}}{\rho_{\text{c}}^{(0)}}$$

$$i = \Lambda, c, b$$

$\rho_{\text{c}}^{(0)}$: Critical density

- In the flat FLRW background, gravitational field equations read

$$H^2 = \frac{\kappa^2}{3}\rho_{\mathrm{eff}} \,, \quad \dot{H} = -\frac{\kappa^2}{2}\left(\rho_{\mathrm{eff}} + p_{\mathrm{eff}}\right)$$

$\rho_{\mathrm{eff}}, \ p_{\mathrm{eff}}$: Effective energy density and pressure from the term $f(R) - R$

$$\rho_{\mathrm{eff}} = \frac{1}{\kappa^2 f'(R)}\left[\frac{1}{2}\left(-f(R) + Rf'(R)\right) - 3H\dot{R}f''(R)\right]$$

$$p_{\mathrm{eff}} = \frac{1}{\kappa^2 f'(R)}\left[\frac{1}{2}\left(f(R) - Rf'(R)\right) + \left(2H\dot{R} + \ddot{R}\right)f''(R) + \dot{R}^2 f'''(R)\right]$$

→ $$\boxed{w_{\mathrm{eff}} = \frac{p_{\mathrm{eff}}}{\rho_{\mathrm{eff}}} = \frac{\left(f(R) - Rf'(R)\right)/2 + \left(2H\dot{R} + \ddot{R}\right)f''(R) + \dot{R}^2 f'''(R)}{\left(-f(R) + Rf'(R)\right)/2 - 3H\dot{R}f''(R)}}$$

- Example: $f(R) = R \underline{- \frac{\mu^{2(n+1)}}{R^n}}$ **[Carroll, Duvvuri, Trodden and Turner, Phys. Rev. D <u>70</u>, 043528 (2004)]**

Second term become important as R decreases.

μ : Mass scale, n : Constant

⇨ $\underline{a \propto t^q}$, $\quad q = \frac{(2n+1)(n+1)}{n+2}$

$$w_{\mathrm{eff}} = -1 + \frac{2(n+2)}{3(2n+1)(n+1)}$$

(For $n = 1$, $\underline{q = 2 \text{ and } w_{\mathrm{eff}} = -2/3}$.)

If $q > 1$, accelerated expansion can be realized.